ECE- 314
SIGNALS AND SYSTEMS
Summer 2013

Laplace Transform

- ✓ Definition
- ✓ Region of Convergence
- ✓ Inverse Laplace Transform
- ✓ Properties of Laplace Transform

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- Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

- Where $s = \sigma + j\omega$

- Note: $X(s)\big|_{s=j\omega} = X(j\omega): FT\ of\ x(t)$

- Convergence: $\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt < \infty$

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- Region of Convergence (ROC) for Laplace Transforms:
  - Values of s for which the Laplace Transform converges.
- Properties:
  - ROC of X(s): made of strips parallel to the jω-axis in the s-plane.
  - If X(s) is rational, then the ROC does not contain any poles.
  - If x(t) is of finite duration, and if we find at least one value of s for which X(s) converges, then the ROC is the entire s-plane.
  - If x(t) is right-sided and if the line $Re\{s\} = \sigma_0$ is in the ROC, then, all values for which $Re\{s\} > \sigma_0$ will be also in the ROC.
  - If x(t) is left-sided and if the line $Re\{s\} = \sigma_0$ is in the ROC, then, all values for which $Re\{s\} < \sigma_0$ will be also in the ROC.
  - If x(t) is two-sided and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC is a strip in the s-plane that includes the line $Re\{s\} = \sigma_0$.

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- Inverse Laplace Transform:

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$x(t)e^{-\sigma t} \xleftrightarrow{FT} \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = X(\sigma + j\omega) = X(s)$$

$$x(t)e^{-\sigma t} = FT^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

$$\rightarrow x(t) = \frac{1}{2\pi}\int_{\sigma + -\infty}^{\sigma + \infty} X(s)e^{st}ds, s = \sigma + j\omega \rightarrow ds = jd\omega$$

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- Properties of Laplace Transform:
  - Linearity:

$$x_1(t) \xleftrightarrow{L} X_1(s), \quad ROC: R_1$$

$$x_2(t) \xleftrightarrow{L} X_2(s), \quad ROC: R_2$$

$$\Rightarrow ax_1(t) + bx_2(t) \xleftrightarrow{L} aX_1(s) + bX_2(s), ROC: includes \quad R_1 \cap R_2$$

  - Time shifting:

$$x(t) \xleftrightarrow{L} X(s), \quad ROC: R$$

$$\Rightarrow x(t - t_0) \xleftrightarrow{L} e^{-st_0} X(s), \quad ROC: R$$

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- Properties of Laplace Transform:
  - Shifting in the s-domain

$$x(t) \xleftrightarrow{L} X(s), \quad ROC: R$$

$$\Rightarrow e^{s_0 t} x(t) \xleftrightarrow{L} X(s - s_0), \quad ROC: R + Re\{s_0\}$$

  - Time-scaling:

$$x(t) \xleftrightarrow{L} X(s), \quad ROC: R$$

$$\Rightarrow x(at) \xleftrightarrow{L} \frac{1}{|a|} X(s/a), \quad ROC: R/a$$

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- Properties of Laplace Transform:
  - Convolution

$$x_1(t) \overset{L}{\longleftrightarrow} X_1(s), \quad ROC: R_1$$

$$x_2(t) \overset{L}{\longleftrightarrow} X_2(s), \quad ROC: R_2$$

$$\Rightarrow x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s)X_2(s), ROC: includes \quad R_1 \cap R_2$$

  - Time differentiation:

$$x(t) \overset{L}{\longleftrightarrow} X(s), \quad ROC: R$$

$$\Rightarrow \frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s), \quad ROC: contains \quad R$$

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- Properties of Laplace Transform:
  - Differentiation in the s-domain:

$$x(t) \xleftrightarrow{L} X(s), \quad ROC: R$$

$$\Rightarrow -tx(t) \xleftrightarrow{L} \frac{dX(s)}{ds}, \quad ROC: R$$

  - Integration

$$x(t) \xleftrightarrow{L} X(s), \quad ROC: R$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{L} \frac{1}{s} X(s), \quad ROC: contains\ R \cap \{Re\{s\} > 0\}$$