Solutions - Midterm Exam

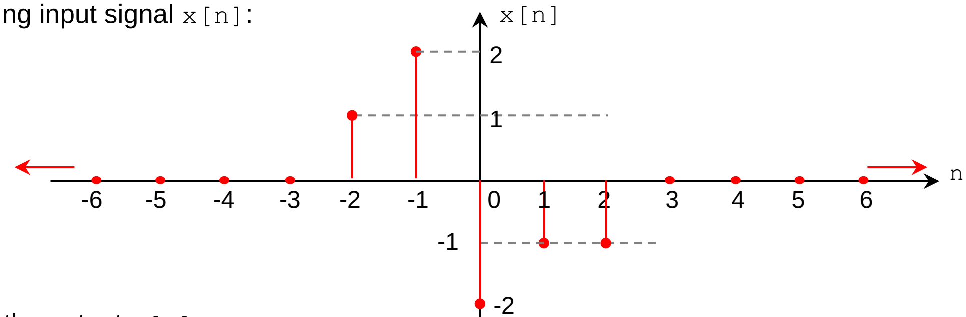
PROBLEM 1 (15 PTS)

Given the following LTI system:

$$y[n] = 2x[n] + x[n-1] + 2x[n-3]$$

a) Sketch the impulse response h[n].

b) Given the following input signal x[n]:



Carefully sketch the output y[n].

c) Identify and find the proper Fourier representation (DTFS, FS, FT, or DTFT) of x[n] and h[n].

d) Find the frequency response of the output y[n].

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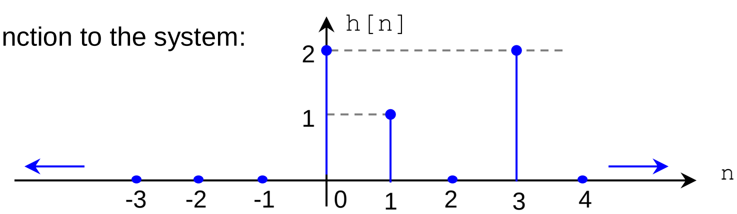
a) Impulse response: We apply the delta function to the system:

$$y[n] = 2x[n] + x[n-1] + 2x[n-3]$$

$$h[n] = 2\delta[n] + \delta[n-1] + 2\delta[n-3]$$



b) $y[n] = x[n] * h[n]$

x[n]	1	2	-2	-1	-1
	-2	-1	0	1	2
x[k]	1	2	-2	-1	-1
	-2	-1	0	1	2

h[n]	2	1	0	2
	0	1	2	3
h[-k+n]	2	0	1	2
	n-3	n-2	n-1	n

n = -2: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[-2] = 2**

n = -1: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[-1] = 5**

n = 0: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[0] = -2**

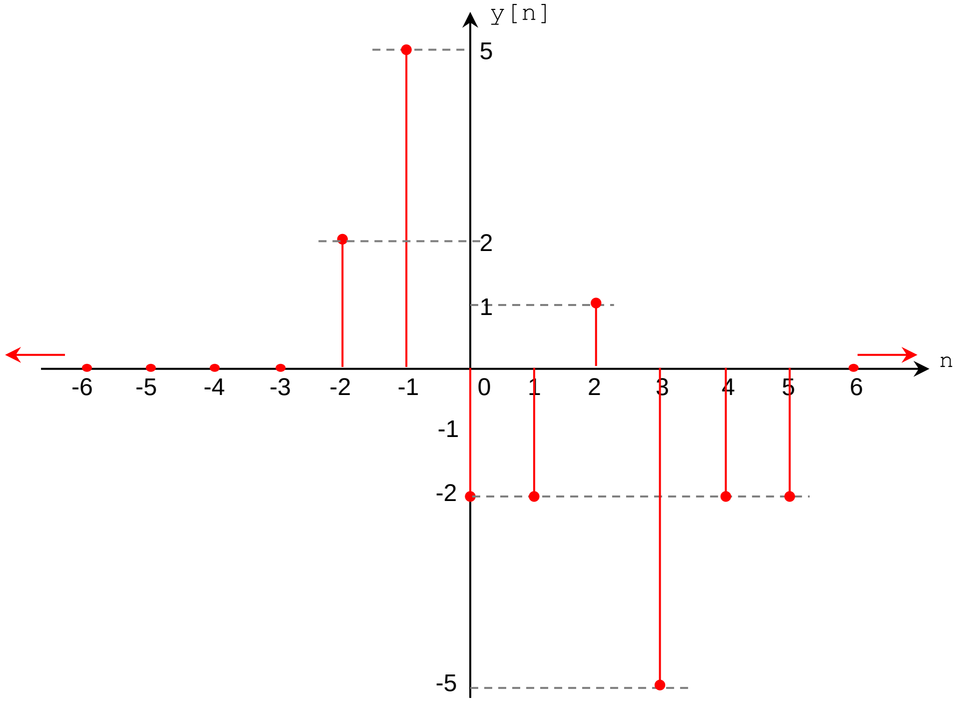
n = 1: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[1] = -2**

n = 2: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[2] = 1**

n = 3: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[3] = -5**

n = 4: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[4] = -2**

n = 5: x[k] = [1, 2, -2, -1, -1] at k = -2, -1, 0, 1, 2; h[-k+n] = [2, 0, 1, 2] at n-3, n-2, n-1, n **=> y[5] = -2**



c) $x[n], h[n]$ are not periodic $\rightarrow$ DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-2}^{2} x[n]e^{-j\Omega n} = e^{2j\Omega} + 2e^{j\Omega} - 2 - e^{-j\Omega} - e^{-2j\Omega}$$

$$X(e^{j\Omega}) = e^{j\Omega}(e^{2} - e^{-2} + 2 - e^{-1}) - 2$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{n=0}^{3} h[n]e^{-j\Omega n} = 2 + e^{-j\Omega} + 2e^{-3j\Omega}$$

d) $Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega})$

$X(e^{j\Omega}) = \left[e^{j\Omega}(e^{2} - e^{-2} + 2 - e^{-1}) - 2\right][2 + e^{-j\Omega} + 2e^{-3j\Omega}]$

PROBLEM 2 (15 PTS)

a) Obtain the FT for the following signal: $x(t) = e^{-at}u(t)$

b) Use the properties of Fourier representation (e.g., time-differentiation, convolution, time-shift, frequency-shift) to find the FT of:

$$y(t) = 2\frac{d}{dt}\{e^{-4t}u(t) * e^{-t}u(t-3)\},$$ '*' denotes convolution.

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a) $x(t) = e^{-at}u(t)$

$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$, converges for a > 0

$$X(j\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{(a+j\omega)}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

b) $w(t) = e^{-4t}u(t), v(t) = e^{-t}u(t-3),\ z(t) = w(t) * v(t)$

$$x(t) = 2\frac{d}{dt}z(t)$$

Differentiation Property of FT: If $z(t) \xleftrightarrow{FT} Z(j\omega)$, then: $\frac{d}{dt}z(t) \xleftrightarrow{FT} j\omega Z(j\omega)$

Then: $X(j\omega) = 2j\omega Z(j\omega)$

Convolution Property of FT: $z(t) = w(t) * v(t) \xleftrightarrow{FT} Z(j\omega) = W(j\omega)V(j\omega)$

Then: $X(j\omega) = 2j\omega W(j\omega)V(j\omega)$

Knowledge of a common FT pair: $s(t) = e^{-at}u(t) \xleftrightarrow{FT} S(j\omega) = \frac{1}{a+j\omega}$
Then, if $w(t) = e^{-4t}u(t)$, then: $W(j\omega) = \frac{1}{4+j\omega}$

Now: $v(t) = e^{-t}u(t-3) = e^{-3}e^{-(t-3)}u(t-3) = e^{-3}r(t-3)$,
where: $r(t) = e^{-t}u(t)$, and $R(j\omega) = \frac{1}{1+j\omega}$

Time Shift Property of FT: If $r(t) \xleftrightarrow{FT} R(j\omega)$, then: $r(t-t0) \xleftrightarrow{FT} e^{-j\omega t0}R(j\omega)$
Then: $V(j\omega) = e^{-3}e^{-j3\omega}R(j\omega) = \frac{e^{-3}e^{-j3\omega}}{1+j\omega}$

Finally:

$$X(j\omega) = 2j\omega W(j\omega)V(j\omega) = \frac{2j\omega e^{-3}e^{-j3\omega}}{(1+j\omega)(4+j\omega)}$$

PROBLEM 3 (10 PTS)

Given the following system: $y[n] = \alpha^n x[n], \alpha > 0$

a) Determine whether the system is i) memoryless, ii) causal, iii) linear, and iv) time-invariant. Justify your answers.
b) If $x[n] = \delta[n]$, then evaluate $y[n]$.
c) Let's call your response in (b): $hp[n] = y[n]$.
Is $hp[n] = h[n]$? In other words, can we evaluate the output of the system in response to any input $x[n]$ using convolution: $y[n] = x[n] * h[n]$? Yes or no? Why?

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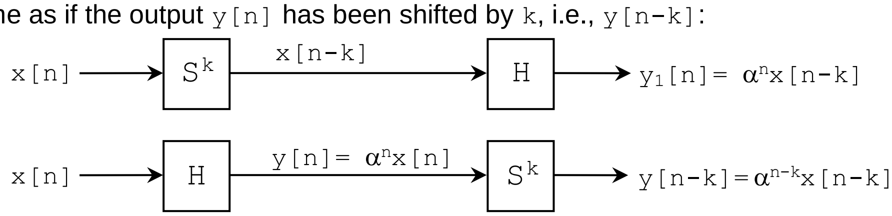
a) The system is memoryless: It does not depend on future samples nor past samples.
The system is causal: It does not depend on future samples.

Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a,b, are real numbers, then the output should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $ax_A[n] + bx_B[n]$, then:
$y[n] = \alpha^n(ax_A[n] + bx_B[n]) = \alpha^n(ax_{A[}n]) + \alpha^n(bx_B[n])$

Now: $ay_A[n] + by_B[n] = a\alpha^n x_A[n]u[n] + b\alpha^n x_B[n]$
We see that $y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

Time invariance:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by k, i.e., $y[n-k]$:



Response of system to a shifted input $x[n-k]$: $y_1[n] = \alpha^n x[n-k]$
Output $y[n]$ shifted by k: $y[n-k] = \alpha^{n-k} x[n-k]$
We see that $y[n-k] \neq y_1[n]$. Thus, the system is NOT time invariant.

b) If $x[n] = \delta[n]$, then $y[n] = \alpha^n\delta[n] = \delta[n],\ \alpha > 0$

c) $hp[n] = \delta[n]$. While this is the impulse response of the system, the system is NOT linear an time invariant, therefore we CANNOT apply the convolution of an input with the impulse response in order to get the output.

PROBLEM 4 (10 PTS)

a) The output of a discrete-time system is related to its input `x[n]` as follows:

$$y[n] = ax[n] + bx[n+2] + e^{nc}x[n-1] - x[n-1]$$

where a, b, c, are real values

i. Under what condition (if any) of the values a, b, c is the system causal?
ii. Under what condition (if any) of the values a, b, c is the system memoryless?

b) An LTI system is described by the following impulse response:

$$h[n] = e^{-2n}u[n-2]$$

i. Determine whether the system is i) memoryless, ii) causal, and iii) stable.
ii. If $y[n] = h[n]/2$, what is the input $x[n]$? Provide the equation of $x[n]$.

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a)

i. For the system to be causal, we need b = 0, so that:
$y[n] = ax[n] + e^{nc}x[n-1] - x[n-1]$.

ii. For the system to be memoryless, we need: b = 0, c = 0, so that:
$y[n] = ax[n]$.

b)

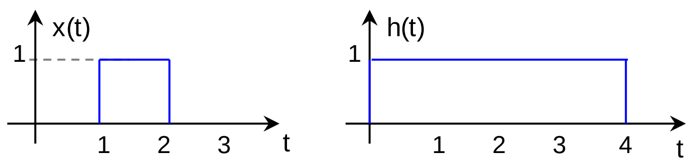
i. The system is NOT <u>memoryless</u>: $h[n] \neq c\delta[n]$.
The system is <u>causal</u>: $h[n] = 0,\ for\ n < 0$
The system is <u>stable</u>:

$$\sum_{-\infty}^{\infty}|h[n]| = \sum_{2}^{\infty}|e^{-2n}| = \sum_{0}^{\infty}e^{-2n} - \sum_{0}^{1}e^{-2n} = \frac{1}{1-e^{-2}} - 1 - e^{-2} = \frac{e^{-4}}{1-e^{-2}} < \infty$$

ii. $x[n] = {}^{\delta[n]}/_{2}$

PROBLEM 5 (20 PTS)

The impulse response of an LTI system and an input signal `x(t)` are depicted below:



a) Obtain the output `y(t)` (or carefully sketch it).
b) Identify and obtain the proper Fourier representation (DTFS, FS, FT, or DTFT) of `x(t)` and `h(t)`. Do not forget to specify the Fourier representation values when the frequency variable is 0.
c) Obtain the frequency response of the output `y(t)`. Do not forget to specify the Fourier representation value when the frequency variable is 0.

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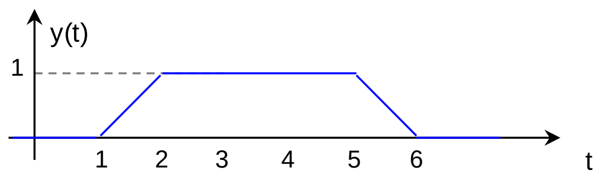
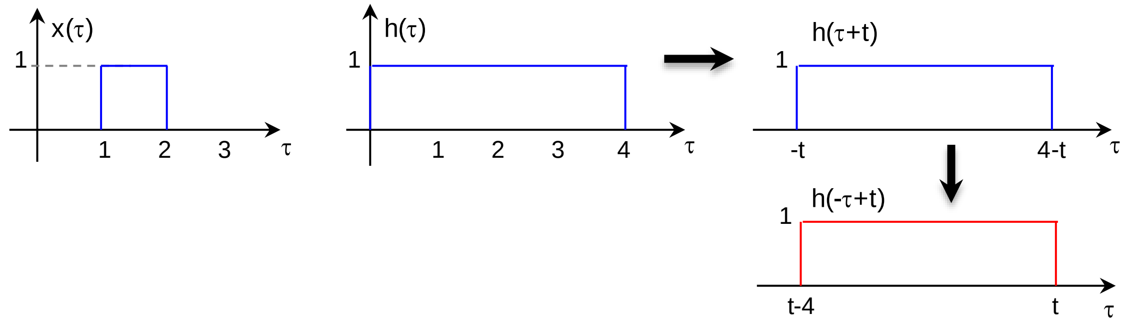
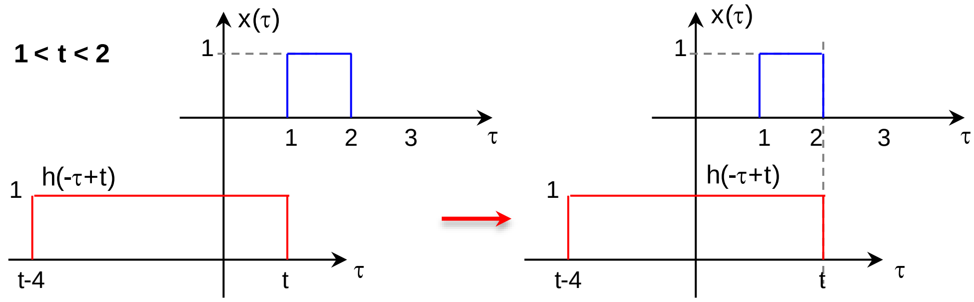
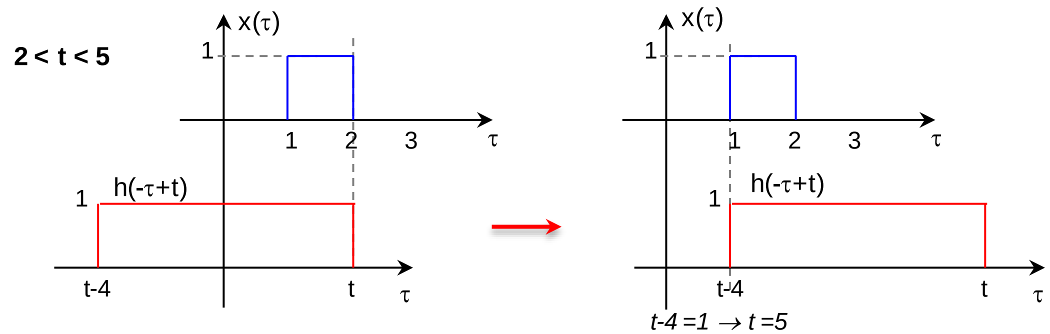
a) CT convolution:

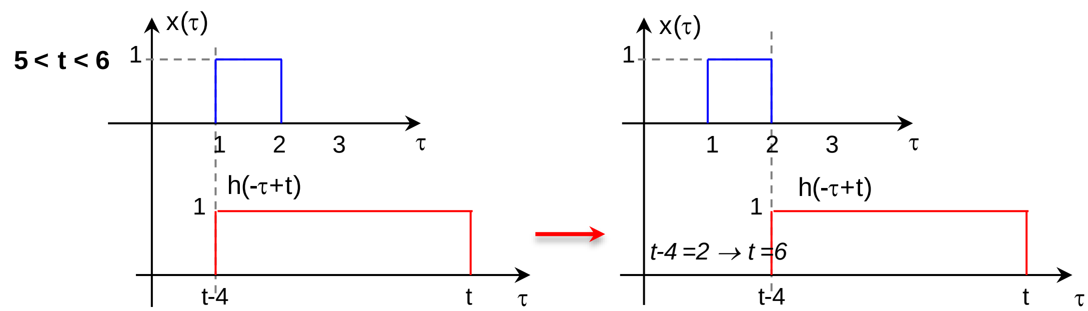
- $1 < t < 2$: $w_t(\tau) = 1,\ 1 < \tau < t,\ \rightarrow\ y(t) = \int_1^t 1d\tau = t - 1,\ 1 < t < 2$
- $2 < t < 5$: $w_t(\tau) = 1,\ 1 < \tau < 2,\ \rightarrow\ y(t) = \int_1^2 1d\tau = 1,\ 2 < t < 5$
- $5 < t < 6$: $w_t(\tau) = 1,\ t-4 < \tau < 2,\ \rightarrow\ y(t) = \int_{t-4}^2 1d\tau = 6 - t,\ 5 < t < 6$
- $t < 1\ or\ t > 6$: $w_t(\tau) = 0,\ \forall\tau,\ \rightarrow\ y(t) = 0,\ t < 1\ or\ t > 6$











b) $x(t), h(t)$ nonperiodic → FT.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{1}^{2} e^{-j\omega t}dt$$

$$X(j\omega) = -\frac{1}{j\omega}e^{-j\omega t}\Big|_1^2 = \frac{1}{j\omega}(e^{-j\omega} - e^{-2j\omega}), \omega \neq 0; \quad X(j\omega) = \int_{1}^{2} 1dt = 1, \omega = 0$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{0}^{4} e^{-j\omega t}dt$$

$$H(j\omega) = -\frac{1}{j\omega}e^{-j\omega t}\Big|_0^4 = \frac{1}{j\omega}(1 - e^{-4j\omega}), \omega \neq 0; \quad H(j\omega) = \int_{0}^{4} 1dt = 4, \omega = 0$$

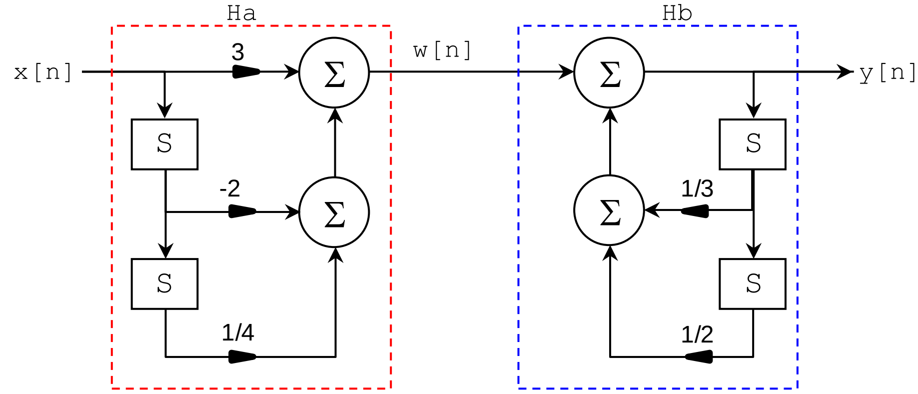
c)

$$Y(j\omega) = H(j\omega)X(j\omega) = 4, \omega = 0$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{j\omega}(e^{-j\omega} - e^{-2j\omega})\frac{1}{j\omega}(1 - e^{-4j\omega}) = \frac{(1 - e^{-4j\omega})(e^{-2j\omega} - e^{-j\omega})}{\omega^2}, \omega \neq 0$$

PROBLEM 6 (10 PTS)

The following representation of an LTI system (called H) is called Direct Form I. We can think of the system as the cascade of two systems Ha and Hb, each with `ha[n]` and `hb[n]` as impulse responses.



a) Get `y[n]` in terms of `x[n]` and past samples of `y[n]`.

b) The entire system H relates `y[n]` to `x[n]`, and its impulse response is `h[n]`. How would you express `h[n]` in terms of `ha[n]` and `hb[n]`?

c) Draw the Direct Form II representation of the system H. Write down the procedure.

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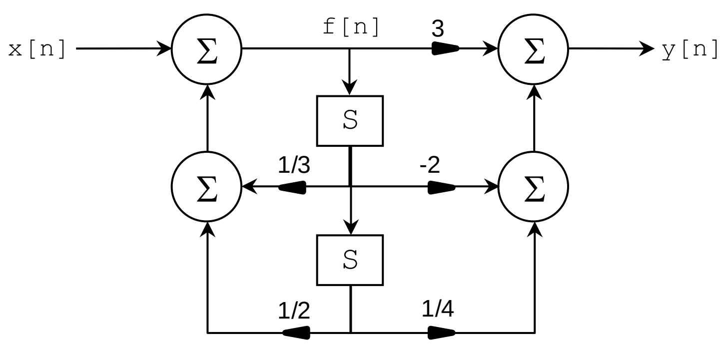
a) H1: `w[n] = 3x[n] - 2x[n-1] + 0.25x[n-2]`
H2: `y[n] = (1/3)y[n-1] + 0.5y[n-2] + w[n]`
`y[n] = (1/3)y[n-1] + 0.5y[n-2] + 3x[n] - 2x[n-1] + 0.25x[n-2]`

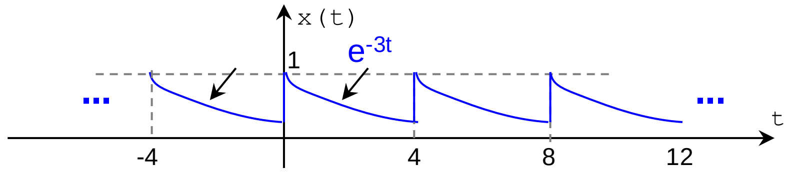
b) $h[n] = h_a[n] * h_b[n]$

c) H2: `f[n] = x[n] + (1/3)f[n-1] + 0.5y[n-2]`
H1: `y[n] = 3f[n] - 2f[n-1] + 0.25f[n-2]`



PROBLEM 7 (20 PTS)

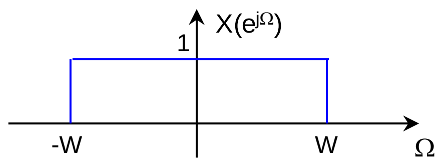
a) For the following periodic signal `x(t)`, identify and get the proper Fourier representation (it should only depend on the frequency variable). Is the Fourier representation periodic? If so, what is the period?



b) For the following signal `x[n]`, identify and get the proper Fourier representation (it should only depend on the frequency variable). Is the Fourier representation periodic? If so, what is the period?

$$x[n] = \sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right)$$

c) You are provided with the DTFT of a signal `x[n]`. Assume $W < \pi$. Find the signal `x[n]`. Do not forget to specify the value of `x[n]` when `n=0`. What is the period of $X(e^{j\Omega})$?



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a) `x(t)`, is periodic (T=4) → FS. $\omega = {}^{2\pi}/_{T} = {}^{\pi}/_{2}$

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega t}dt = \frac{1}{4}\int_0^4 e^{-3t}e^{-jk\omega t}dt = \frac{1}{4}\int_0^4 e^{-(3+jk\omega)t}dt = -\frac{1}{4}\left(\frac{1}{3+jk\omega}\right)e^{-(3+jk\omega)t}\Big|_0^4$$

$$X[k] = -\frac{1}{4}\left(\frac{1}{3+jk\omega}\right)\left(e^{-(3+jk\omega)4} - 1\right) = \frac{1}{4}\left(\frac{1}{3+jk\frac{\pi}{2}}\right)\left(1 - e^{-\left(3+jk\frac{\pi}{2}\right)4}\right)$$

$$X[k] = \frac{1}{4}\left(\frac{1}{3+jk\frac{\pi}{2}}\right)\left(1 - e^{-12-jk2\pi}\right) = \frac{1}{4}\left(\frac{1}{3+jk\frac{\pi}{2}}\right)\left(1 - e^{-12}\right), \qquad X[k]\ is\ non-periodic$$

b) $N = \frac{2\pi m}{\frac{\pi}{3}} = 6m, \rightarrow N = 6,\ \Omega = {}^{2\pi}/_{N} = {}^{\pi}/_{3}$. `x[n]` is periodic → DTFS

We rewrite the signal:

$$x[n] = sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right) = \frac{e^{j\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)} - e^{-j\left(\frac{\pi}{3}n-\frac{\pi}{2}\right)}}{2j} = \frac{e^{j\frac{\pi}{2}}}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) = \frac{1}{2}e^{j\frac{\pi}{3}n} - \frac{1}{2}e^{-j\frac{\pi}{3}n}$$

If we remember the equation: $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega n} = \sum_{k=0}^{N-1} X[k]e^{jk\frac{\pi}{3}n}$

We notice that: $X[1] = \frac{1}{2}, X[-1] = -\frac{1}{2}$

The Fourier representation is periodic (`N=6` samples), and it is given by:

$$X[-1] = -\frac{1}{2}, \qquad X[0] = 0, \qquad X[1] = \frac{1}{2}, \qquad X[2] = 0, \qquad X[3] = 0, \qquad X[4] = 0$$

<u>Alternative method</u> (applying the DTFS formula):

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right)e^{-jk\Omega n} = \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{e^{j\left(\frac{\pi}{3}n+\frac{\pi}{2}\right)} - e^{-j\left(\frac{\pi}{3}n-\frac{\pi}{2}\right)}}{2j}\right)e^{-jk\Omega n}$$

$$X[k] = \frac{e^{j\frac{\pi}{2}}}{2jN}\sum_{n=0}^{N-1}\left(e^{j\left(\frac{\pi}{3}n\right)} - e^{-j\left(\frac{\pi}{3}n\right)}\right)e^{-jk\Omega n} = \frac{1}{2N}\sum_{n=0}^{N-1} e^{j\left(\frac{\pi}{3}-k\frac{\pi}{3}\right)n} - e^{-j\left(\frac{\pi}{3}+k\frac{\pi}{3}\right)n}$$

$$X[k] = \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{\pi}{3}(k-1)n} - e^{-j\frac{\pi}{3}(k+1)n}$$

$k = 1$:

$$X[k] = \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{\pi}{3}(k-1)n} - e^{-j\frac{\pi}{3}(k+1)n} = \frac{1}{2N}\sum_{n=0}^{N-1} 1 - e^{-j\frac{\pi}{3}2n} = \frac{1}{2N}\left(N + \sum_{n=0}^{N-1} e^{-j\frac{\pi}{3}2n}\right)$$

$$X[k] = \frac{1}{2N}\left(N + \frac{1-e^{-j\frac{\pi}{3}2N}}{1-e^{-j\frac{\pi}{3}2}}\right) = \frac{1}{12}\left(6 + \frac{1-e^{-j4\pi}}{1-e^{-j\frac{\pi}{3}2}}\right) = \frac{1}{2}, k = 1$$

$k = -1$:

$$X[k] = \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{\pi}{3}(k-1)n} - e^{-j\frac{\pi}{3}(k+1)n} = \frac{1}{2N}\sum_{n=0}^{N-1}\left(e^{j\frac{\pi}{3}2n} - 1\right)$$

$$X[k] = \frac{1}{2N}\left(\frac{1-e^{j\frac{\pi}{3}2N}}{1-e^{j\frac{\pi}{3}2}} - N\right) = \frac{1}{12}\left(\frac{1-e^{j4\pi}}{1-e^{j\frac{\pi}{3}2}} - 6\right) = -\frac{1}{2}, k = -1$$

$k \neq \pm 1:, i.e., k = 0,2,3,4$

$$X[k] = \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{\pi}{3}(k-1)n} - e^{-j\frac{\pi}{3}(k+1)n} = \frac{1}{2N}\left(\frac{1-e^{-j\frac{\pi}{3}(k-1)N}}{1-e^{-j\frac{\pi}{3}(k-1)}} - \frac{1-e^{-j\frac{\pi}{3}(k+1)N}}{1-e^{-j\frac{\pi}{3}(k+1)}}\right)$$

$$X[k] = \frac{1}{12}\left(\frac{1-e^{-j2\pi(k-1)}}{1-e^{-j\frac{\pi}{3}(k-1)}} - \frac{1-e^{-j2\pi(k+1)}}{1-e^{-j\frac{\pi}{3}(k+1)}}\right) = 0$$

c) You are provided with the DTFT of a signal `x[n]`. Assume $W < \pi$. Find the signal `x[n]`. Do not forget to specify the value of `x[n]` when `n=0`. What is the period of $X(e^{j\Omega})$?

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}d\Omega$$

$n = 0$:

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = \frac{W}{\pi}$$

$n \neq 0$:

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}d\Omega = \frac{1}{2\pi jn}e^{j\Omega n}\Big|_{-W}^{W} = \frac{1}{2\pi jn}\left(e^{jnW} - e^{-jnW}\right) = \frac{1}{\pi n}sin(Wn)$$

The DTFT $X(e^{j\Omega})$ is periodic with period 2π.

BONUS PROBLEM (+ 15 PTS)

The output of an LTI system in response to an input $x[n] = \delta[n-p]$ is $y[n] = \alpha^n u[n-1]$.
Note that p is an integer number, and $0 < \alpha < 1$

Find the frequency response of the impulse response, as well as the impulse response of this system.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n-p]e^{-j\Omega n} = e^{-j\Omega p}, \qquad \delta[n-p] = 1 \; when \; n = p$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n-1]e^{-j\Omega n} = \sum_{n=1}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=1}^{\infty} (\alpha e^{-j\Omega})^n = \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n - 1 = \frac{1}{1-\alpha e^{-j\Omega}} - 1$$

$$Y(e^{j\Omega}) = \frac{\alpha e^{-j\Omega}}{1-\alpha e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = e^{j\Omega p}\frac{\alpha e^{-j\Omega}}{1-\alpha e^{-j\Omega}}$$

Time Shift Property of DTFT: $y[n+p] \xleftrightarrow{DTFT} e^{j\Omega p}Y(e^{j\Omega})$

Therefore: $h[n] = y[n+p] = \alpha^{n+p}u[n+p-1]$

Alternative way:

With the time shift property of the DTFT and the knowledge of the DTFT of the impulse function, we can demonstrate that:

$$\delta[n+p] \xleftrightarrow{DTFT} e^{j\Omega p}$$

Then, given that: $H(e^{j\Omega}) = e^{j\Omega p}Y(e^{j\Omega})$
We can use convolution property to express the impulse response as:

$$h[n] = \delta[n+p] * \alpha^n u[n-1]$$

By performing this convolution, we find that:
$h[n] = \alpha^{n+p}, n \geq 1-p$. This is the same as $h[n] = \alpha^{n+p}u[n+p-1]$

Verifying that our result $h[n]$ is correct:

$$H(e^{j\Omega}) = \sum_{n=1-p}^{\infty} \alpha^{n+p}e^{-j\Omega n} = \alpha^p \sum_{n=1-p}^{\infty} (\alpha e^{-j\Omega})^n$$

With the formula: $\sum_{n=k}^{\infty} b^n = \frac{b^k}{1-b}, |b| < 1$, we then demonstrate that:

$$H(e^{j\Omega}) = \alpha^p \frac{(\alpha e^{-j\Omega})^{1-p}}{1-\alpha e^{-j\Omega}} = e^{j\Omega p}\frac{\alpha e^{-j\Omega}}{1-\alpha e^{-j\Omega}}$$