Solutions - Homework # 6

PROBLEM 1

Determine the Laplace transform, the associated region of convergence, and the pole-zero plot for each of the following signals:

a) $x(t) = te^{at}u(-t)$

b) $x(t) = t^2e^{-at}u(-t)$

c) $x(t) = t^2e^{-at}u(t) + te^{-at}u(-t) + 3u(t) + \delta(t)$

d) $x(t) = e^{-2(t+1)}u(t+1) + 2e^{-(t+4)}u(-t-4)$

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a) $x(t) = te^{at}u(-t)$

We know:

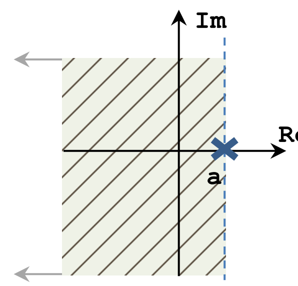
$$-e^{at}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s-a}, ROC: Re\{s\} < a$$

Applying differentiation property: $*\frac{d}{ds}(s+b)^n = n(s+b)^{n-1}$

$$(-t)\{-e^{at}u(-t)\} \overset{L}{\longleftrightarrow} \frac{d}{ds}\left(\frac{1}{s-a}\right) = -\frac{1}{(s-a)^2}, ROC: Re\{s\} < a$$

Finally:

$$x(t) = te^{at}u(-t) \overset{L}{\longleftrightarrow} X(s) = -\frac{1}{(s-a)^2}, ROC: Re\{s\} < a$$



b) $x(t) = t^2e^{-at}u(-t)$

We know (using the result from (a)):

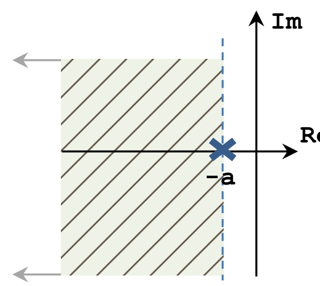
$$te^{-at}u(-t) \overset{L}{\longleftrightarrow} X(s) = -\frac{1}{(s+a)^2}, ROC: Re\{s\} < -a$$

Applying differentiation property:

$$-t\left(te^{-at}u(-t)\right) \overset{L}{\longleftrightarrow} -\frac{d}{ds}\left(\frac{1}{(s+a)^2}\right) = \frac{2}{(s+a)^3}, ROC: Re\{s\} < -a$$

Finally:

$$x(t) = t^2e^{-at}u(-t) \overset{L}{\longleftrightarrow} X(s) = -\frac{2}{(s+a)^3}, ROC: Re\{s\} < -a$$



c) $x(t) = t^2e^{-at}u(t) + te^{-at}u(-t) + 3u(t) + \delta(t)$

We know:

$$te^{-at}u(-t) \overset{L}{\longleftrightarrow} X(s) = -\frac{1}{(s+a)^2}, ROC: Re\{s\} < -a$$

$$u(t) \overset{L}{\longleftrightarrow} \frac{1}{s}, Re\{s\} > 0$$

$$\delta(t) \overset{L}{\longleftrightarrow} 1, ROC: \forall s$$

Also:

$$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, ROC: Re\{s\} > -a$$

$$te^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{(s+a)^2}, ROC: Re\{s\} > -a$$

$$t^2e^{-at}u(t) \overset{L}{\longleftrightarrow} X(s) = \frac{2}{(s+a)^3}, ROC: Re\{s\} > -a$$

We notice that ROCs the individual terms do not intersect. Therefore, the Laplace Transform $X(s)$ does not exist.

d) $x(t) = e^{-2(t+1)}u(t+1) + 2e^{-(t+4)}u(-t-4)$

We know:

$$e^{-2t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, ROC: Re\{s\} > -2$$
$$e^{-t}u(-t) \stackrel{L}{\longleftrightarrow} -\frac{1}{s+1}, ROC: Re\{s\} < -1$$

Time-shift property:

$$e^{-2(t+1)}u(t+1) \stackrel{L}{\longleftrightarrow} \frac{e^s}{s+2}, ROC: Re\{s\} > -2$$
$$e^{-(t+4)}u(t+4) \stackrel{L}{\longleftrightarrow} -\frac{e^{4s}}{s+1}, ROC: Re\{s\} < -1$$

Then:

$$X(s) = \frac{e^s}{s+2} - \frac{2e^{4s}}{s+1}, -2 < Re\{s\} < -1$$

Poles and zeros:

$$X(s) = \frac{e^s}{s+2} - \frac{e^{4s}}{s+1} = \frac{e^s(s+1) - 2e^{4s}(s+2)}{(s+2)(s+1)}$$

Poles: $s = -2, s = -1$

Zeros:

$$e^s(s+1) - 2e^{4s}(s+2) = 0 \rightarrow e^s(s+1-2se^{3s} - 4e^{3s}) = 0$$
$$\rightarrow 2(s+2)e^{3s} - (s+1) = 0$$

This is a numerical problem that can be solved with MATLAB®.

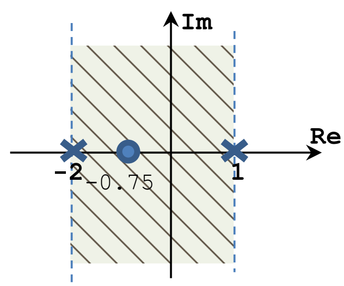
MATLAB code:

```
clear all; close all; clc;
a = 2; b = 4; c = 3; d = -1; e = -1;
sol = fsolve(@(s) myexpfun(s, a, b, c, d, e),0);
```

In a different file ('myexpfun.m'):

```
function F = myexpfun(s, a, b, c, d, e)
    F = (a*s+b)*exp(c*s) + (d*s+e);
end
```

The solution (the zero) is $s = -0.7504$.



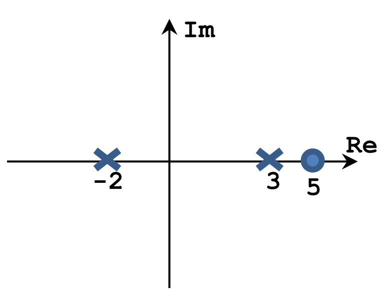
PROBLEM 2

For the following transfer function $H(s)$ of an LTI system:

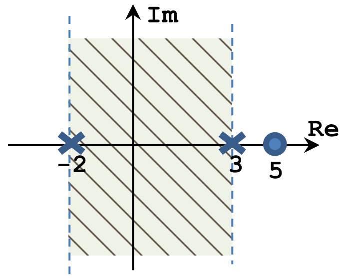
$$H(s) = \frac{s-5}{(s+2)(s-3)}, ROC = ?$$

- Sketch the pole-zero plot.
- If the system is stable, determine the largest possible ROC. Is the system causal? Yes/no? Why?
- If the system is causal, determine the largest possible ROC. Is the system stable? Yes/no? Why?

- Pole-zero plot:

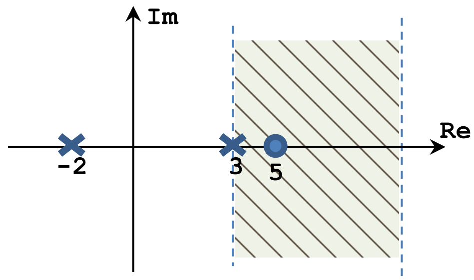


- If $H(s)$ is stable → ROC includes $\sigma = 0$. The largest possible ROC would be: $-2 < Re\{s\} < 3$.



The ROC is not to the right of the rightmost pole, therefore the system is not causal.

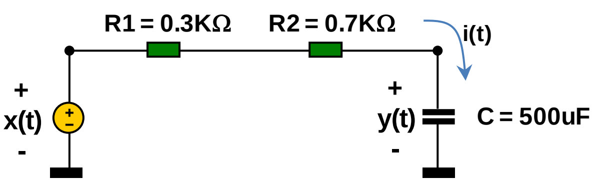
- If $H(s)$ is causal → $h[n]$ is right-sided → ROC is to the right of rightmost pole.
The largest possible ROC would be: $Re > 3$.

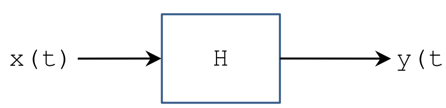


The ROC does not include $\sigma = 0$ (the axis $j\omega$ is not included). Thus, the system is not stable.

PROBLEM 3

Given the following LTI system:





$x(t) = \frac{2}{5}e^{-3t}u(t), \qquad y(0^+) = 4$

a) Determine the differential equation that relates $x(t)$ and $y(t)$.

$$x(t) = i(t)R1 + i(t)R2 + y(t), \qquad i(t) = C\frac{dy(t)}{dt}$$

b) Determine the Laplace Transform of the input signal $x(t)$ with the associated region of convergence.
c) Determine the Laplace Transform of the output signal $y(t)$ with the associated region of convergence. Sketch the pole-zero plot.
d) Determine the output signal $y(t)$.

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a) Differential equation:

$$x(t) = C\frac{dy(t)}{dt}R1 + C\frac{dy(t)}{dt}R2 + y(t) = (R1 + R2)C\frac{dy(t)}{dt} + y(t) = 0.5\frac{dy(t)}{dt} + y(t)$$

$$\rightarrow \frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

b) Unilateral Laplace Transform: It is the same as the Bilateral Laplace Transform when $x(t) = 0, t < 0$.

$$x(t) = e^{-at}u(t) \overset{L}{\longleftrightarrow} X_u(s) = \frac{1}{s+a}, Re\{s\} > -a$$

Then:

$$x(t) = \frac{2}{5}e^{-3t}u(t) \overset{L}{\longleftrightarrow} X_u(s) = \frac{2}{5(s+3)}, Re\{s\} > -3$$

c) Unilateral Laplace Transform of $y(t)$: Taking Unilateral Laplace Transform to both sides of the differential equation and applying the differentiation property:

$$sY_u(s) - y(0^+) + 2Y_u(s) = 2X_u(s) \rightarrow Y_u(s)(s+2) = 2X_u(s) + y(0^+)$$

$$\rightarrow Y_u(s) = \frac{1}{s+2}\left(2X_u(s) + y(0^+)\right) = \frac{1}{s+2}\left(\frac{4}{5(s+3)} + 4\right)$$

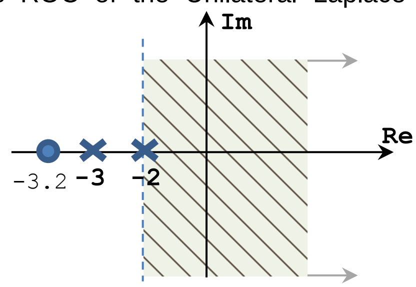
$$Y_u(s) = \frac{4}{5(s+2)(s+3)} + \frac{4}{(s+2)} = \frac{4+20(s+3)}{5(s+2)(s+3)} = \frac{4(5s+16)}{5(s+2)(s+3)}$$

$$Y_u(s) = \frac{4}{5(s+2)(s+3)} + \frac{4}{(s+2)} = \frac{24}{5(s+2)} - \frac{4}{5(s+3)}$$

Unilateral Laplace Transform: It is the bilateral Laplace Transform of a signal whose values for $t < 0^+$ have been set to zero. This is, the signal is right-sided. Thus ROC of the Unilateral Laplace Transform is located to the right.

$$\rightarrow \frac{1}{s+2}, Re\{s\} > -2; \qquad \frac{1}{s+3}, Re\{s\} > -3$$

→ ROC of $Y_u(s)$: $(Re\{s\} > -2) \cap (Re\{s\} > -3) = Re\{s\} > -2$



d) Output signal $y(t)$. Since we know the ROCs of $\frac{1}{s+2}$ and $\frac{1}{s+3}$, we can quickly determine the time-domain output signal:

$$y(t) = \frac{24}{5}e^{-2t}u(t) - \frac{4}{5}e^{-3t}u(t)$$

PROBLEM 4

Determine the Z-transform, the ROC, and the pole-zero plot for each of the following signals. Also, determine whether the DTFT exists for each signal.

a) $x[n] = -2a^n u[n-1], \ a > 1$

b) $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]$

c) $x[n] = u[n] + \left(\frac{1}{4}\right)^{|n|}$

d) $x[n] = a^{|n|} + 2\delta[n], \ 0 < a < 1$

e) $x[n] = a^{|n|} + 2\delta[n], \ a > 1$

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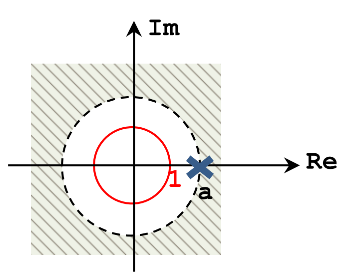
a) $x[n] = -2a^n u[n-1], \ a > 1$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -2\sum_{n=1}^{\infty} a^n z^{-n} = -2\left(\sum_{n=0}^{\infty} a^n z^{-n} - 1\right) = 2 - 2\sum_{n=0}^{\infty}(az^{-1})^n$$

$$\rightarrow |az^{-1}| < 1 \rightarrow |z| > |a|$$

$$\rightarrow X(z) = 2 - 2\frac{1}{1-az^{-1}} = 2 - \frac{2z}{z-a} = -\frac{2a}{z-a}, |z| > |a|$$

The ROC does not include the unit circle, therefore the DTFT does not exist.



b) $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]$
We know:

$$\left(\frac{1}{4}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}}, ROC: |z| > \frac{1}{4}$$

$$\left(\frac{1}{4}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} -\frac{z}{z-\frac{1}{4}}, ROC: |z| < \frac{1}{4}$$

The individual ROCs do not intersect, therefore $X(z)$ does not exist.

c) $x[n] = u[n] + \left(\frac{1}{4}\right)^{|n|}$
We know:

$$x[n] = u[n] \overset{z}{\longleftrightarrow} X(z) = \frac{z}{z-1}, ROC: |z| > 1$$

Now: $v[n] = \left(\frac{1}{4}\right)^{|n|} = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^{-n} u[-n-1] = \left(\frac{1}{4}\right)^n u[n] + 4^n u[-n-1]$
We know:

$$\left(\frac{1}{4}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}}, ROC: |z| > \frac{1}{4}$$

$$4^n u[-n-1] \overset{z}{\longleftrightarrow} -\frac{z}{z-4}, ROC: |z| < 4$$

Then:

$$V(z) = \frac{z}{z-\frac{1}{4}} - \frac{z}{z-4}, \quad \frac{1}{4} < |z| < 4$$

Finally:

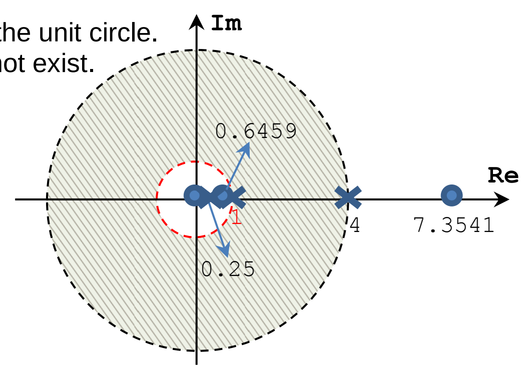
$$X(z) = \frac{z}{z-1} + \frac{z}{z-\frac{1}{4}} - \frac{z}{z-4}, \quad 1 < |z| < 4$$

Poles and zeros:

$$X(z) = \frac{z}{z-1} + \frac{z}{z-\frac{1}{4}} - \frac{z}{z-4} = \frac{z\left(z^2 - 8z + \frac{19}{4}\right)}{(z-1)\left(z-\frac{1}{4}\right)(z-4)}$$

$$X(z) = \frac{z(z-7.3541)(z-0.6459)}{(z-1)\left(z-\frac{1}{4}\right)(z-4)}, 1 < |z| < 4$$

The ROC does not include the unit circle.
Therefore, the DTFT does not exist.



d) $x[n] = a^{|n|} + 2\delta[n], \ 0 < a < 1$
We know:

$$\delta[n] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1, \forall z$$

Now: $v[n] = a^{|n|} = a^n u[n] + a^{-n} u[-n-1] = a^n u[n] + (a^{-1})^n u[-n-1]$
We know:

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-a}, ROC: |z| > a$$

$$(a^{-1})^n u[-n-1] \overset{z}{\longleftrightarrow} -\frac{z}{z-\frac{1}{a}}, ROC: |z| < \frac{1}{a}$$

Then:

$$V(z) = \frac{z}{z-a} - \frac{z}{z-\frac{1}{a}}, \qquad a < |z| < \frac{1}{a}$$

Finally:

$$X(z) = 2 + \frac{z}{z-a} - \frac{z}{z-\frac{1}{a}}, \qquad a < |z| < \frac{1}{a}$$

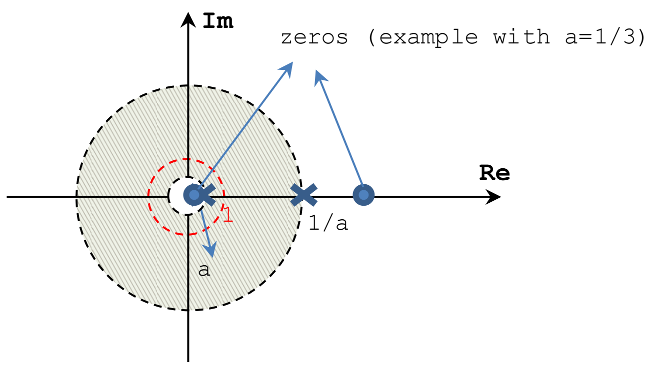
If $0 < a < 1$, then $a < |z| < \frac{1}{a}$ is not empty. Thus, $X(z)$ exists.

$$\rightarrow X(z) = 2 + \frac{z}{z-a} - \frac{z}{z-\frac{1}{a}} = \frac{2z^2 - z\left(a+\frac{3}{a}\right)+2}{(z-a)\left(z-\frac{1}{a}\right)}, a < |z| < \frac{1}{a}$$

Poles: $z = a, z = \frac{1}{a}$

Zeros: $z_{1,2} = \frac{a+\frac{3}{a} \pm \sqrt{\left(a+\frac{3}{a}\right)^2 - 16}}{4}$

The ROC includes the unit circle.
Thus, the DTFT exists.



e) $x[n] = a^{|n|} + 2\delta[n],\ a > 1$
This case is similar to the case of (d). However, $a > 1$.
This means that the ROC $a < |z| < \frac{1}{a}$ is empty. Thus, $X(z)$ does not exist.

PROBLEM 5

For the following Z-Transform, determine the time-domain signal $x[n]$.

a) $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, |z| > \frac{1}{2}$

b) $X(z) = \frac{4-\frac{13}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}$

  i. ROC: $|z| > \frac{1}{2}$
  ii. ROC: $|z| < \frac{1}{8}$
  iii. ROC: $\frac{1}{8} < |z| < \frac{1}{2}$

c) $X(z) = log_2\left(1+\frac{1}{2}z^{-1}\right), |z| < \frac{1}{2}$

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a) $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, |z| > \frac{1}{2}$

Tip: To decompose in partial fractions, use $x = z^{-1}$

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = -\frac{3}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+\frac{1}{2}z^{-1}} = -\frac{3z}{z+\frac{1}{4}} + \frac{4z}{z+\frac{1}{2}}, |z| > \frac{1}{2}$$

$\frac{z}{z+\frac{1}{4}}$: It must include $|z| > \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z+\frac{1}{4}}$:

$$\left(-\frac{1}{4}\right)^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{z}{z+\frac{1}{4}}, ROC: |z| > \frac{1}{4}$$

$$-\left(-\frac{1}{4}\right)^n u[-n-1] \stackrel{Z}{\longleftrightarrow} \frac{z}{z+\frac{1}{4}}, ROC: |z| < \frac{1}{4}$$

Thus, the ROC of $\frac{z}{z+\frac{1}{4}}$ is $|z| > \frac{1}{4}$.

$\frac{z}{z+\frac{1}{2}}$: It must include $|z| > \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z+\frac{1}{2}}$:

$$\left(-\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{1}{2}}, ROC: |z| > \frac{1}{2}$$

$$-\left(-\frac{1}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Thus, the ROC of $\frac{z}{z+\frac{1}{2}}$ is $|z| > \frac{1}{2}$.

Finally:

$$x[n] = -3\left(-\frac{1}{4}\right)^n u[n] + 4\left(-\frac{1}{2}\right)^n u[n]$$

b) $X(z) = \frac{4-\frac{13}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}$

$$X(z) = \frac{4-\frac{13}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{3}{1-\frac{1}{8}z^{-1}} = \frac{z}{z-\frac{1}{2}} + \frac{3z}{z-\frac{1}{8}}$$

i. ROC: $|z| > \frac{1}{2}$

$\frac{z}{z-\frac{1}{2}}$: It must include $|z| > \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{2}}$:

$$\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| > \frac{1}{2}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{2}}$ is $|z| > \frac{1}{2}$.

$\frac{z}{z-\frac{1}{8}}$: It must include $|z| > \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{8}}$:

$$\left(\frac{1}{8}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| > \frac{1}{8}$$

$$-\left(\frac{1}{8}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| < \frac{1}{8}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{8}}$ is $|z| > \frac{1}{8}$.

Finally:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 3\left(\frac{1}{8}\right)^n u[n]$$

ii. ROC: $|z| < \frac{1}{8}$

$\frac{z}{z-\frac{1}{2}}$: It must include $|z| < \frac{1}{8}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{2}}$:

$$\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| > \frac{1}{2}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{2}}$ is $|z| < \frac{1}{2}$.

$\frac{z}{z-\frac{1}{8}}$: It must include $|z| < \frac{1}{8}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{8}}$:

$$\left(\frac{1}{8}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| > \frac{1}{8}$$

$$-\left(\frac{1}{8}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| < \frac{1}{8}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{8}}$ is $|z| < \frac{1}{8}$.

Finally:

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 3\left(\frac{1}{8}\right)^n u[-n-1]$$

iii. ROC: $\frac{1}{8} < |z| < \frac{1}{2}$

$\frac{z}{z-\frac{1}{2}}$: It must include $\frac{1}{8} < |z| < \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{2}}$:

$$\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| > \frac{1}{2}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{2}}$ is $|z| < \frac{1}{2}$.

$\frac{z}{z-\frac{1}{8}}$: It must include $\frac{1}{8} < |z| < \frac{1}{2}$. There are two possible ROCs for $\frac{z}{z-\frac{1}{8}}$:

$$\left(\frac{1}{8}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| > \frac{1}{8}$$

$$-\left(\frac{1}{8}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z-\frac{1}{8}}, ROC: |z| < \frac{1}{8}$$

Thus, the ROC of $\frac{z}{z-\frac{1}{8}}$ is $|z| > \frac{1}{8}$.

Finally:

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 3\left(\frac{1}{8}\right)^n u[n]$$

c) $X(z) = log_2\left(1 + \frac{1}{2}z^{-1}\right) = (log_2 e)ln\left(1 + \frac{1}{2}z^{-1}\right),\ |z| < \frac{1}{2}$

Differentiation in the z-domain:

$$\rightarrow nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = -z(log_2 e)\frac{\frac{1}{2}(-z^{-2})}{1+\frac{1}{2}z^{-1}} = (log_2 e)\frac{\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}} = (log_2 e)\frac{\frac{1}{2}}{z+\frac{1}{2}} = \left(\frac{log_2 e}{2}\right)\frac{1}{z+\frac{1}{2}}$$

Since the Z transform of $nx[n]$ is rational, let's get the expression for $nx[n]$:

We know:

$$-\left(-\frac{1}{2}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{z}{z+\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Time-shift:

$$-\left(-\frac{1}{2}\right)^{n-1} u[-(n-1)-1] \overset{z}{\longleftrightarrow} \frac{zz^{-1}}{z+\frac{1}{2}} = \frac{1}{z+\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

Finally, we have that:

$$nx[n] = -\left(\frac{log_2 e}{2}\right)\left(-\frac{1}{2}\right)^{n-1} u[-n] \overset{z}{\longleftrightarrow} \left(\frac{log_2 e}{2}\right)\frac{1}{z+\frac{1}{2}}, ROC: |z| < \frac{1}{2}$$

$$\therefore x[n] = \frac{-\left(\frac{log_2 e}{2}\right)\left(-\frac{1}{2}\right)^{n-1} u[-n]}{n}, n \neq 0$$

PROBLEM 6

For the following difference equation of an LTI system:

$$y[n] + \frac{3}{2}y[n-1] = 2x[n-1] + x[n]$$

a) Obtain the algebraic expression of Z-Transform of the impulse response of the system, i.e. $H(z)$. Sketch the pole-zero plot.
b) If the system is causal, determine the largest possible ROC. Get the expression of $h[n]$ assuming the largest possible ROC. Is the system stable? Yes or no? Why?
c) If the system is stable, determine the largest possible ROC. Get the expression of $h[n]$ assuming the largest possible ROC. Is the system causal? Yes or no? Why?

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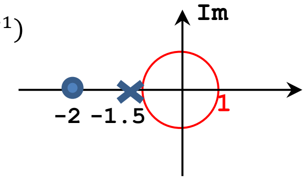
a) Taking the Z-Transform on both sides (also applying the time-shift property):

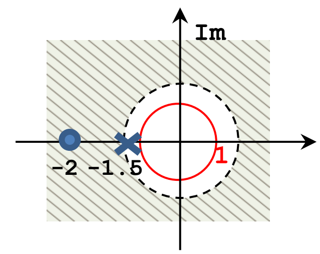
$$Y(z) + \frac{3}{2}z^{-1}Y(z) = 2z^{-1}X(z) + X(z) \rightarrow Y(z)\left(1 + \frac{3}{2}z^{-1}\right) = X(z)(1 + 2z^{-1})$$

$$\rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 + \frac{3}{2}z^{-1}} = \frac{z+2}{z+\frac{3}{2}}$$



b) If the system is causal → $h[n]$ is right-sided → ROC of $H(z)$ is outside of outermost pole.
→ Largest ROC: $|z| > \frac{3}{2}$

$$\rightarrow H(z) = \frac{z}{z+\frac{3}{2}} + \frac{2}{z+\frac{3}{2}}, |z| > \frac{3}{2}$$



$\frac{z}{z+\frac{3}{2}}$: It must include $|z| > \frac{3}{2}$. There are two possible ROCs for $\frac{z}{z+\frac{3}{2}}$:

$$\left(-\frac{3}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{3}{2}}, ROC: |z| > \frac{3}{2}$$

$$-\left(-\frac{3}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{3}{2}}, ROC: |z| < \frac{3}{2}$$

Thus, the ROC of $\frac{z}{z+\frac{3}{2}}$ is $|z| > \frac{3}{2}$. Then, we have: $\left(-\frac{3}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{3}{2}}, ROC: |z| > \frac{3}{2}$

Time-shift property: $\left(-\frac{3}{2}\right)^{n-1} u[n-1] \overset{Z}{\longleftrightarrow} \frac{zz^{-1}}{z+\frac{3}{2}} = \frac{1}{z+\frac{3}{2}}, ROC: |z| > \frac{3}{2}$

Finally:

$$h[n] = \left(-\frac{3}{2}\right)^n u[n] + 2\left(-\frac{3}{2}\right)^{n-1} u[n-1]$$

$|z| > \frac{3}{2}$ does not include the unit circle. Thus, the DTFT does not exist and the system is NOT stable.

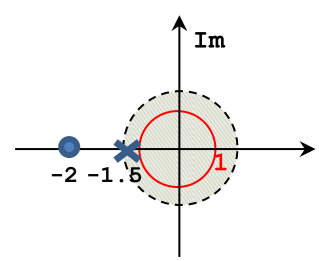
c) If the system is stable, the ROC must include the unit circle.
For $\frac{z}{z+\frac{3}{2}}$, there are only two possible ROCs (the same applies to $\frac{1}{z+\frac{3}{2}}$).

$$\left(-\frac{3}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{3}{2}}, ROC: |z| > \frac{3}{2}$$

$$-\left(-\frac{3}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z+\frac{3}{2}}, ROC: |z| < \frac{3}{2}$$

For the ROC of $H(z)$ to include the unit circle, we must pick $|z| < \frac{3}{2}$.

Time-shift property: $\left(-\frac{3}{2}\right)^{n-1} u[-(n-1)-1] \overset{Z}{\longleftrightarrow} \frac{1}{z+\frac{3}{2}}, ROC: |z| < \frac{3}{2}$



Finally:

$$h[n] = -\left(-\frac{3}{2}\right)^n u[-n-1] - 2\left(-\frac{3}{2}\right)^{n-1} u[-n]$$

$h[n]$ is not right-sided. Also, the ROC is not outside of the outermost pole. Thus, $H(z)$ is NOT causal.