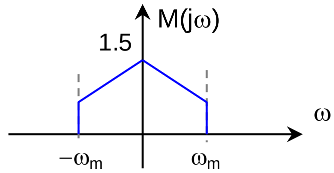
Solutions - Homework # 5

PROBLEM 1

A full Amplitude modulator has the following characteristics:

- Carrier frequency: 3KHz
- Carrier amplitude: 1 (the units are usually given in volts)

The message signal $m(t)$ has the following spectrum:



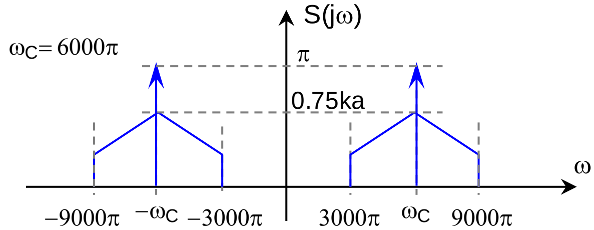
Sketch the frequency response of the modulated signal when:

a) $\omega_m = 3000\pi$
b) $\omega_m = 5000\pi$
c) $\omega_m = 11000\pi$

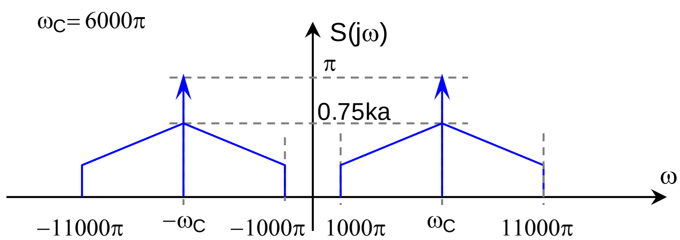
- Is there an instance in which the modulator is not working properly? Explain.
- Is there a minimum value of ω_m for the modulator to work properly?

$\omega_C = 6000\pi, A_C = 1 \rightarrow s(t) = \left(1 + k_a m(t)\right)Cos(6000\pi t)$

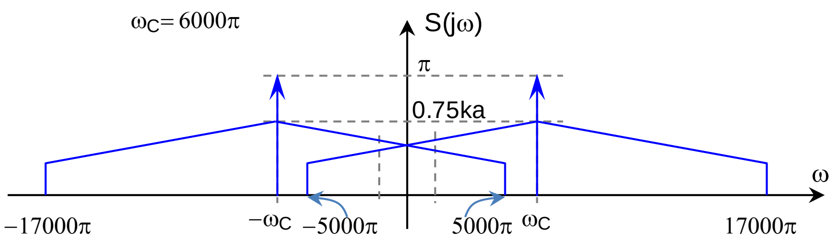
a) $\omega_m = 3000\pi$



b) $\omega_m = 5000\pi$



c) $\omega_m = 11000\pi$



- The case of (c) is not working properly as the spectra are being superimposed.
- For the modulator to work, we need that the spectra are not superimposed.
  This requires that: $\omega_m = \omega_c = 6000\pi$.

PROBLEM 2

Given the following carrier and message signals:

$$m(t) = A_0Cos(\omega_0 t) + A_1Cos(2\omega_0 t)$$

Assume that $\omega_c > 10\omega_0$.

- For full amplitude modulation, provide the expression for the frequency response of the modulated signal. Then sketch the frequency response (magnitude).
- Provide an expression for the percentage of modulation.
  - What is the minimum value of the amplitude sensitivity factor k_a that avoids over-modulation?
  - Provide the value k_a for the following percentages of modulation: 40%, 50%, and 80%

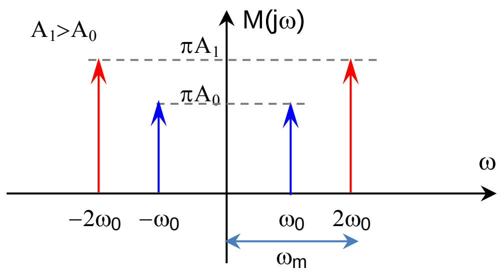
$m(t)$ is a periodic signal with $T = \frac{2\pi}{\omega_0}$. $T = \frac{2\pi}{\omega_0}k = \frac{2\pi}{2\omega_0}r \rightarrow 2k = r \rightarrow k = 1, r = 2 \rightarrow T = \frac{2\pi}{\omega_0}$

Fourier Series:

$$m(t) = A_0Cos(\omega_0 t) + A_1Cos(2\omega_0 t) = \frac{A_0}{2}e^{j\omega_0 t} + \frac{A_0}{2}e^{-j\omega_0 t} + \frac{A_1}{2}e^{j2\omega_0 t} + \frac{A_1}{2}e^{-j2\omega_0 t}$$
$$m(t) = M[1]e^{j\omega_0 t} + M[-1]e^{-j\omega_0 t} + M[2]e^{j2\omega_0 t} + M[-2]e^{-j2\omega_0 t}$$

$$M[k] = \begin{cases} \frac{A_0}{2}, & k = \pm 1 \\ \frac{A_1}{2}, & k = \pm 2 \\ 0, & otherwise \end{cases}$$

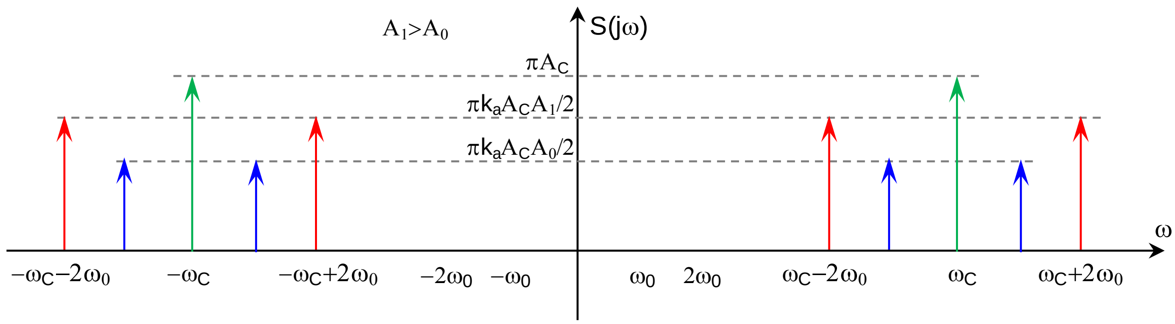
$$M(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} M[k]\delta(\omega - k\omega_0) = \pi A_0\delta(\omega - \omega_0) + \pi A_0\delta(\omega + \omega_0) + \pi A_1\delta(\omega - 2\omega_0) + \pi A_1\delta(\omega + 2\omega_0)$$



Modulated signal:

$$s(t) = (1 + k_a m(t))A_C Cos(\omega_C t) = (1 + k_a\{A_0Cos(\omega_0 t) + A_1Cos(2\omega_0 t)\})A_C Cos(\omega_C t)$$
$$S(j\omega) = \pi A_C[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)] + \frac{k_a A_C}{2}\left[M\big(j(\omega + \omega_C)\big) + M\big(j(\omega - \omega_C)\big)\right]$$

$$\rightarrow S(j\omega) = \pi A_C[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)]$$
$$+ \frac{k_a A_C}{2}[\pi A_0\delta(\omega - \omega_0 - \omega_C) + \pi A_0\delta(\omega + \omega_0 - \omega_C) + \pi A_1\delta(\omega - 2\omega_0 - \omega_C) + \pi A_1\delta(\omega + 2\omega_0 - \omega_C)]$$
$$+ \frac{k_a A_C}{2}[\pi A_0\delta(\omega - \omega_0 + \omega_C) + \pi A_0\delta(\omega + \omega_0 + \omega_C) + \pi A_1\delta(\omega - 2\omega_0 + \omega_C) + \pi A_1\delta(\omega + 2\omega_0 + \omega_C)]$$



- Percentage of modulation:

$$\% \ of \ modulation: max|k_a m(t)| \times 100\%$$
$$\rightarrow max|k_a m(t)| = max|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)|$$

$\rightarrow \frac{d}{dt}\{k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)\} = 0$

$\rightarrow -A_0\omega_0 Sin(\omega_0 t) - 2A_1\omega_0 Sin(2\omega_0 t) = 0$

$\rightarrow -A_0 Sin(\omega_0 t) = 2A_1 Sin(2\omega_0 t) \rightarrow -A_0 Sin(\omega_0 t) = 4A_1 Sin(\omega_0 t) Cos(\omega_0 t), \omega_0 \neq 0$

If $\omega_0 t = \pi k, k \ integer \rightarrow Sin(\omega_0 t) = 0 \rightarrow -A_0 Sin(\omega_0 t) = 4A_1 Sin(\omega_0 t)Cos(\omega_0 t)$

$\rightarrow |k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |-k_a A_0 + k_a A_1| = |k_a A_1 - k_a A_0|$

If we restrict to: $\omega_0 t \neq 2\pi k, k \ integer$:

$\rightarrow |k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |-k_a A_0 - k_a A_1| = |k_a A_1 + k_a A_0|$

In general (since we do not know the signs of A_1 and A_0, we better say that:

$max|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |k_a A_1| + |k_a A_0| = |k_a|(|A_1| + |A_0|)$

If $\omega_0 t \neq \pi k, k \ integer \rightarrow Sin(\omega_0 t) \neq 0$

$\rightarrow -A_0 = 4A_1 Cos(\omega_0 t) \ \rightarrow Cos(\omega_0 t) = -\frac{A_0}{4A_1} \rightarrow t = \frac{1}{\omega_0} Cos^{-1}\left(-\frac{A_0}{4A_1}\right), |A_0| \leq 4|A_1|$

$\rightarrow Cos(2\omega_0 t) = 2Cos^2(\omega_0 t) - 1 = \frac{1}{8}\left(\frac{A_0}{A_1}\right)^2 - 1$

$\rightarrow max|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |k_a| \left|-A_0 \frac{A_0}{4A_1} + A_1\left(\frac{1}{8}\left(\frac{A_0}{A_1}\right)^2 - 1\right)\right|$

$= |k_a| \left|-\frac{A_0^2}{4A_1} + A_1\left(\frac{1}{8}\left(\frac{A_0}{A_1}\right)^2 - 1\right)\right| = |k_a|\left|-\frac{A_0^2}{4A_1} + A_1 + \frac{1}{8}A_1\left(\frac{A_0}{A_1}\right)^2\right| = |k_a|\left|-\frac{A_0^2}{4A_1} + A_1 + \frac{1}{8}\frac{A_0^2}{A_1}\right|$

$= |k_a|\left|-\frac{A_0^2}{8A_1} + A_1\right| = |k_a|\left|-\frac{A_0}{8}\left(\frac{A_0}{A_1}\right) + A_1\right|$

We know that $|A_0| \leq 4|A_1| \rightarrow \left|\frac{A_0}{A_1}\right| \leq 4$

Then, the maximum value of $|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)|$ is given by:

$max|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |k_a|\left|-\frac{A_0}{8}(\pm 4) + A_1\right| = |k_a|\left|\pm\frac{A_0}{2} + A_1\right|$

Finally: Since $|A_1| + |A_0| > \left|\pm\frac{A_0}{2} + A_1\right|$, the percentage of modulation is given by:

$$\rightarrow max|k_a m(t)| = max|k_a A_0 Cos(\omega_0 t) + k_a A_1 Cos(2\omega_0 t)| = |k_a|(|A_1| + |A_0|)$$

To avoid over-modulation:

$$|k_a|(|A_1| + |A_0|) \leq 1$$

Minimum value of k_a:

$$|k_a|(|A_1| + |A_0|) = 1 \rightarrow |k_a| = \frac{1}{|A_1| + |A_0|}$$

For percentage of modulation = 40%:

$$|k_a|(|A_1| + |A_0|) = 0.4 \rightarrow |k_a| = \frac{0.4}{|A_1| + |A_0|}$$

For percentage of modulation = 50%:

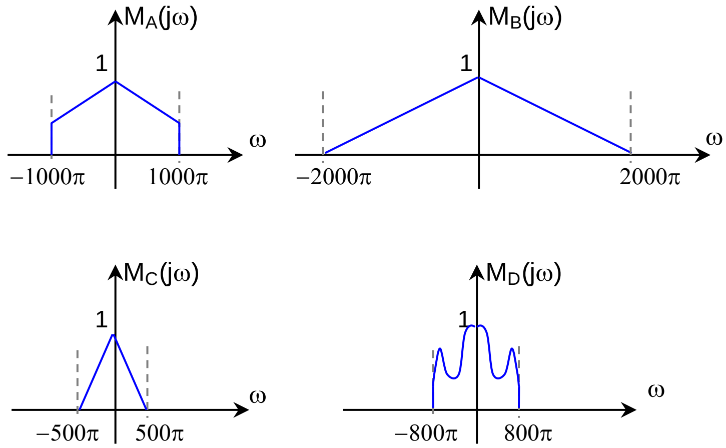
$$|k_a|(|A_1| + |A_0|) = 0.5 \rightarrow |k_a| = \frac{0.5}{|A_1| + |A_0|}$$

For percentage of modulation = 80%:

$$|k_a|(|A_1| + |A_0|) = 0.8 \rightarrow |k_a| = \frac{0.8}{|A_1| + |A_0|}$$

PROBLEM 3

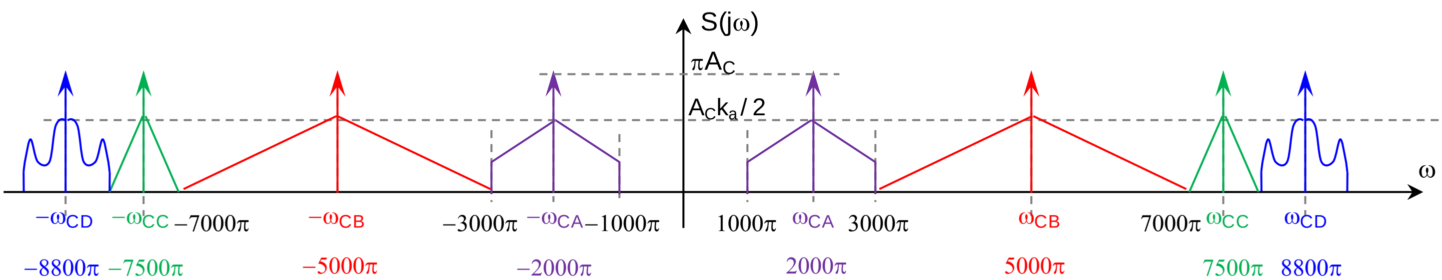
The following are the spectra of four messages:



We want to transmit these messages over a communication channel using full amplitude modulation, where the modulated signals are added in the time-domain. We do not want the spectrums to overlap each other, lest the messages can be damaged.

It is clear that we need four carriers of different frequencies. Provide the frequencies of the carriers (in Hz) that allow for proper communication. Sketch the frequency response of the composite of the four modulated signals.

There are many solutions to this problem.



- Carrier A: $\omega_{CA} = 2000\pi \rightarrow f_{CA} = 1KHz$
- Carrier B: $\omega_{CB} = 5000\pi \rightarrow f_{CB} = 2.5KHz$
- Carrier C: $\omega_{CC} = 7500\pi \rightarrow f_{CC} = 3.75KHz$
- Carrier D: $\omega_{CD} = 8800\pi \rightarrow f_{CD} = 4.4KHz$

PROBLEM 4

Determine the Laplace transform, the associated region of convergence for each of the following signals. Sketch the pole-zero plot for each of the following signals:

a) $x(t) = -e^{at}u(-t), a > 0$

b) $x(t) = e^{-at}u(t), a < 0$

c) $x(t) = Cos\left(\frac{\pi}{7}t\right)[u(t) - u(t-10)]$

d) $x(t) = 3u(t-4)$

e) $x(t) = e^{-3t}u(t) + 2e^{-4t}u(-t)$

f) $x(t) = \delta(t) + e^{-a|t|}$

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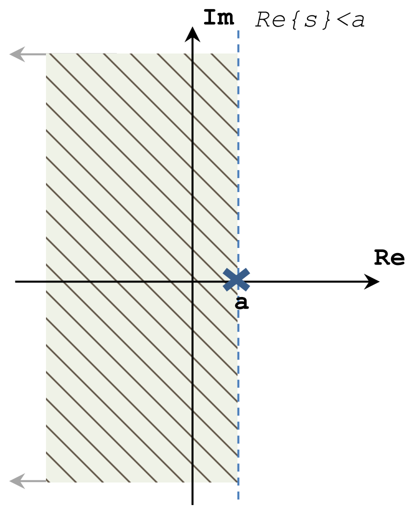
a) $x(t) = -e^{at}u(-t), a > 0$

We know:

$$x(t) = -e^{-at}u(-t) \overset{L}{\longleftrightarrow} X(s) = \frac{1}{s+a}, ROC: Re\{s\} < -a$$

Then:

$$X(s) = \frac{1}{s-a}, ROC: Re\{s\} < a$$



Using the definition:

$$X(s) = -\int_{-\infty}^{0} e^{at}e^{-st}dt = \frac{1}{s-a}e^{(a-\sigma-j\omega)t}\Big|_{-\infty}^{0}$$

$$X(s) = \frac{1}{s-a}\left(e^{(a-\sigma-j\omega)0} - e^{(a-\sigma-j\omega)-\infty}\right), \qquad e^{(j\omega)(\pm\infty)} \leq 1$$

$X(s)$ is integrable if $a - \sigma > 0 \rightarrow Re\{s\} < a$

$$\rightarrow X(s) = \frac{1}{s-a}, Re\{s\} < a$$

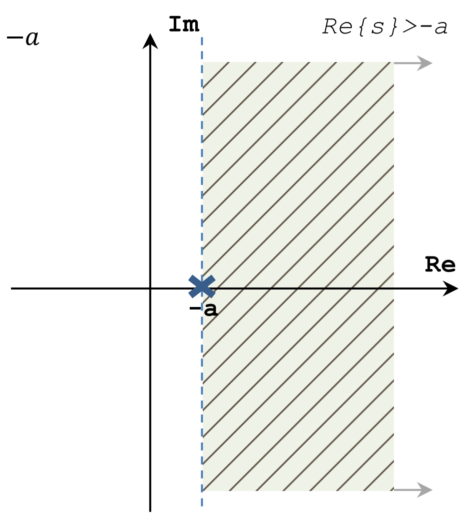
b) $x(t) = e^{-at}u(t), a < 0$

We know:

$$x(t) = e^{-at}u(t) \overset{L}{\longleftrightarrow} X(s) = \frac{1}{s+a}, ROC: Re\{s\} > -a$$

Then:

$$X(s) = \frac{1}{s+a}, ROC: Re\{s\} > -a$$



c) $x(t) = Cos\left(\frac{\pi}{7}t\right)[u(t) - u(t-10)]$

$$X(s) = \frac{1}{2}\int_0^{10}\left(e^{-j\frac{\pi}{7}t} + e^{j\frac{\pi}{7}t}\right)e^{-st}dt = \frac{1}{2}\left\{\int_0^{10} e^{-\left(s-j\frac{\pi}{7}\right)t}dt + \int_0^{10} e^{-\left(s+j\frac{\pi}{7}\right)t}dt\right\}$$

$$X(s) = \frac{1}{2}\left\{-\frac{1}{\left(s-j\frac{\pi}{7}\right)}e^{-\left(s-j\frac{\pi}{7}\right)t}\Big|_0^{10} - \frac{1}{\left(s+j\frac{\pi}{7}\right)}e^{-\left(s+j\frac{\pi}{7}\right)t}\Big|_0^{10}\right\}$$

$$X(s) = \frac{1}{2}\left\{-\frac{1}{\left(s-j\frac{\pi}{7}\right)}\left(e^{-10\left(s-j\frac{\pi}{7}\right)} - 1\right) - \frac{1}{\left(s+j\frac{\pi}{7}\right)}\left(e^{-10\left(s+j\frac{\pi}{7}\right)} - 1\right)\right\}$$

$$X(s) = \frac{1}{2}\left\{\frac{1}{\left(s-j\frac{\pi}{7}\right)}\left(1 - e^{-10\left(s-j\frac{\pi}{7}\right)}\right) + \frac{1}{\left(s+j\frac{\pi}{7}\right)}\left(1 - e^{-10\left(s+j\frac{\pi}{7}\right)}\right)\right\}, s \neq \pm j\frac{\pi}{7}$$

$s = j\frac{\pi}{7}$:

$$X(s) = \frac{1}{2}\left\{\int_0^{10} e^{-\left(s-j\frac{\pi}{7}\right)t}dt + \int_0^{10} e^{-\left(s+j\frac{\pi}{7}\right)t}dt\right\} = \frac{1}{2}\left\{10 + \int_0^{10} e^{-\left(j\frac{\pi}{7}+j\frac{\pi}{7}\right)t}dt\right\} = \frac{1}{2}\left\{10 - \frac{1}{j\frac{2\pi}{7}}\left(e^{-j\frac{20\pi}{7}} - 1\right)\right\}$$

$s = -j\frac{\pi}{7}$:

$$X(s) = \frac{1}{2}\left\{\int_0^{10} e^{-\left(s-j\frac{\pi}{7}\right)t}dt + \int_0^{10} e^{-\left(s+j\frac{\pi}{7}\right)t}dt\right\} = \frac{1}{2}\left\{\int_0^{10} e^{-\left(-j\frac{\pi}{7}-j\frac{\pi}{7}\right)t}dt + 10\right\} = \frac{1}{2}\left\{\frac{1}{j\frac{20\pi}{7}}\left(e^{j\frac{20\pi}{7}} - 1\right) + 10\right\}$$

$\rightarrow X(s)$ converges for all s. Then, the ROC is the entire s-plane.

<u>Poles and zeros</u>**:**

$$X(s) = \frac{1}{2}\left\{\frac{1}{\left(s-j\frac{\pi}{7}\right)}\left(e^{-10\left(s-j\frac{\pi}{7}\right)} - 1\right) + \frac{1}{\left(s+j\frac{\pi}{7}\right)}\left(e^{-10\left(s+j\frac{\pi}{7}\right)} - 1\right)\right\}, s \neq \pm j\frac{\pi}{7}$$

$$\rightarrow \left(e^{-10\left(s-j\frac{\pi}{7}\right)} - 1\right)\left(s+j\frac{\pi}{7}\right) + \left(e^{-10\left(s+j\frac{\pi}{7}\right)} - 1\right)\left(s-j\frac{\pi}{7}\right) = 0$$

Notice that $s = \pm j\frac{\pi}{7}$ are zeros. But they cancel out with the poles $s = \pm j\frac{\pi}{7}$. Thus, there are no poles.

$$se^{-10\left(s-j\frac{\pi}{7}\right)} - s + j\frac{\pi}{7}e^{-10\left(s-j\frac{\pi}{7}\right)} - j\frac{\pi}{7} + se^{-10\left(s+j\frac{\pi}{7}\right)} - s - j\frac{\pi}{7}e^{-10\left(s+j\frac{\pi}{7}\right)} + j\frac{\pi}{7} = 0$$

$$se^{-10\left(s-j\frac{\pi}{7}\right)} + se^{-10\left(s+j\frac{\pi}{7}\right)} + j\frac{\pi}{7}e^{-10\left(s-j\frac{\pi}{7}\right)} - j\frac{\pi}{7}e^{-10\left(s+j\frac{\pi}{7}\right)} - 2s = 0$$

$$se^{-10s}\left(e^{j\frac{10\pi}{7}} + e^{-j\frac{10\pi}{7}}\right) + j\frac{\pi}{7}e^{-10s}\left(e^{j\frac{10\pi}{7}} - e^{-j\frac{10\pi}{7}}\right) - 2s = 0$$

$$2se^{-10s}Cos\left(\frac{10\pi}{7}\right) - \frac{2\pi}{7}e^{-10s}Sin\left(\frac{10\pi}{7}\right) - 2s = 0$$

$$e^{-10s}\left(2sCos\left(\frac{10\pi}{7}\right) - \frac{2\pi}{7}Sin\left(\frac{10\pi}{7}\right)\right) - 2s = 0$$

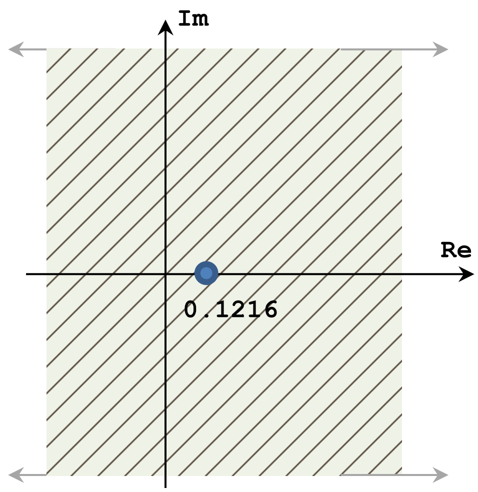
This is a numerical problem that can be solved with MATLAB®.
MATLAB code:

```
clear all; close all; clc;
a = 2*cos(10*pi/7); b = -(2*pi/7)*sin(10*pi/7);
c = -10; d = -2; e = 0;
sol = fsolve(@(s) myexpfun(s, a, b, c, d, e),0);
```

In a different file ('myexpfun.m'):

```
function F = myexpfun(s, a, b, c, d, e)
    F = (a*s+b)*exp(c*s) + (d*s+e);
end
```

The solution (the zero) is $s = 0.1216$.



d) $x(t) = 3u(t-4)$
We know:

$$x(t) = e^{-at}u(t) \xleftrightarrow{L} X(s) = \frac{1}{s+a}, ROC: Re\{s\} > -a$$

$$\rightarrow x(t) = u(t) \xleftrightarrow{L} X(s) = \frac{1}{s}, ROC: Re\{s\} > 0$$

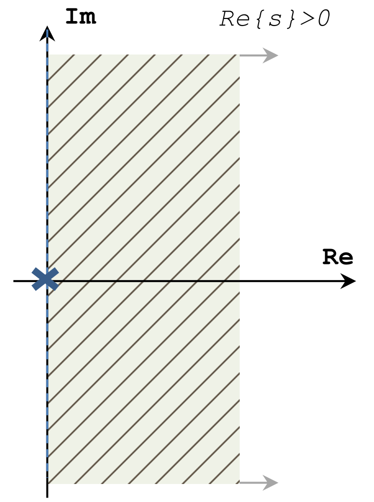
Time-shifting:

$$\rightarrow x(t-t_0) \xleftrightarrow{L} e^{-st_0}X(s), ROC: Re\{s\} > 0$$

Now: If $x(t) = 3u(t-4)$:

$$\rightarrow x(t) = 3u(t-4) \xleftrightarrow{L} X(s) = \frac{3e^{-4s}}{s}, ROC: Re\{s\} > 0$$

There is a pole at $s = 0$. No zeros.



Using the definition:

$$X(s) = 3\int_4^{\infty} e^{-st}dt = \frac{1}{s}e^{-(\sigma+j\omega)t}\Big|_0^{\infty} = \frac{3}{s}\left(e^{-(\sigma+j\omega)4} - e^{-(\sigma+j\omega)\infty}\right),$$

$X(s)$ is integrable if $\sigma > 0 \rightarrow Re\{s\} > 0$

$$\rightarrow X(s) = \frac{3e^{-4s}}{s}, Re\{s\} > 0$$

e) $x(t) = e^{-3t}u(t) + 2e^{-4t}u(-t)$
We know:

$$e^{-3t}u(t) \xleftrightarrow{L} \frac{1}{s+3}, ROC: Re\{s\} > -3$$

$$e^{-4t}u(-t) \xleftrightarrow{L} -\frac{1}{s+4}, ROC: Re\{s\} < -4$$

Next, we perform a linear combination of these two results. However, the intersection of the individual ROCs is empty, therefore the Laplace transform of $x(t)$ does not exist.

f) $x(t) = \delta(t) + e^{-a|t|}$

$$\delta(t) \xleftrightarrow{L} \int_{-\infty}^{\infty} \delta(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1, \qquad e^{-(\sigma+j\omega)0} = 1$$

$v(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$

We know:

$$e^{-at}u(t) \xleftrightarrow{L} \frac{1}{s+a}, ROC: Re\{s\} > -a$$

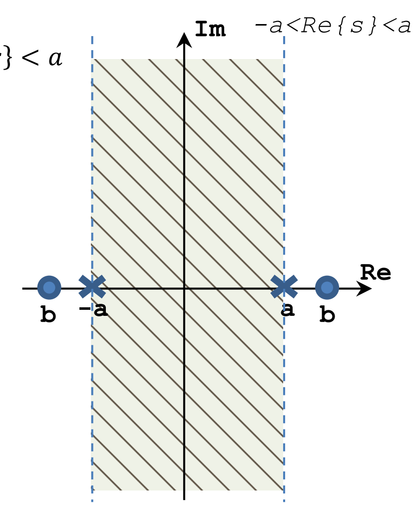
$$e^{at}u(-t) \xleftrightarrow{L} -\frac{1}{s-a}, ROC: Re\{s\} < a$$

Then:

$$X(s) = 1 + \frac{1}{s+a} - \frac{1}{s-a}, -a < Re\{s\} < a$$

If a < 0, then there is no ROC and $X(s)$ does not exist.
If a > 0, then the ROC exists.

$$X(s) = 1 + \frac{1}{s+a} - \frac{1}{s-a}, -a < Re\{s\} < a, a > 0$$



Poles and zeros:

$$X(s) = 1 - \frac{2a}{(s+a)(s-a)} = \frac{s^2 - a^2 - 2a}{(s+a)(s-a)}$$

$$X(s) = \frac{(s+b)(s-b)}{(s+a)(s-a)}, b = \sqrt{a^2 + 2a}$$

PROBLEM 5

Determine the time function $x(t)$ for the following Laplace transforms with their associated regions of convergence.

a) $X(s) = \frac{1}{s+1}, Re\{s\} > -1$
b) $X(s) = \frac{s}{s^2+7s+10}, -5 < Re\{s\} < -2$
c) $X(s) = \frac{s+1}{s^2+5s+6}, Re\{s\} > -2$
d) $X(s) = \frac{s+1}{s^2+5s+6}, Re\{s\} < -3$
e) $X(s) = \frac{1}{(s+3)(s+4)}, -4 < Re\{s\} < -3$

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a) $X(s) = \frac{1}{s+1}, Re\{s\} > -1$

$$e^{-at}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+a}, Re\{s\} > -a$$
$$\rightarrow x(t) = e^{-t}u(t), Re\{s\} > -1$$

b) $X(s) = \frac{s}{s^2+7s+10}, -5 < Re\{s\} < -2$

$$X(s) = \frac{s}{s^2+7s+10} = \frac{s}{(s+2)(s+5)} = \frac{A}{(s+2)} + \frac{B}{(s+5)} = \frac{s(A+B)+5A+2B}{(s+2)(s+5)}$$
$$\rightarrow A + B = 1, 5A + 2B = 0 \rightarrow A = -\frac{2}{3}, B = \frac{5}{3}$$
$$X(s) = -\frac{2}{3(s+2)} + \frac{5}{3(s+5)}, -5 < Re\{s\} < -2$$

$\frac{1}{s+2}$: It must include $-5 < Re\{s\} < -2$. There are two possible ROCs for $\frac{1}{s+2}$:

$$e^{-2t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, Re\{s\} > -2$$
$$-e^{-2t}u(-t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, Re\{s\} < -2$$

Thus, the ROC of $\frac{1}{s+2}$ is $Re\{s\} < -2$

$\frac{1}{s+5}$: It must include $-5 < Re\{s\} < -2$. There are two possible ROCs for $\frac{1}{s+5}$:

$$e^{-5t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+5}, Re\{s\} > -5$$
$$-e^{-5t}u(-t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+5}, Re\{s\} < -5$$

Thus, the ROC of $\frac{1}{s+5}$ is $Re\{s\} > -5$.

Finally:

$$x(t) = \frac{2}{3}e^{-2t}u(-t) + \frac{5}{3}e^{-5t}u(t)$$

c) $X(s) = \frac{s+1}{s^2+5s+6}, Re\{s\} > -2$

$$X(s) = \frac{s+1}{s^2+5s+6} = \frac{s}{(s+2)(s+3)} = -\frac{1}{(s+2)} + \frac{2}{(s+3)}, Re\{s\} > -2$$

$\frac{1}{s+2}$: It must include $Re\{s\} > -2$. There are two possible ROCs for $\frac{1}{s+2}$:

$$e^{-2t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, Re\{s\} > -2$$
$$-e^{-2t}u(-t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, Re\{s\} < -2$$

Thus, the ROC of $\frac{1}{s+2}$ is $Re\{s\} > -2$

$\frac{1}{s+3}$: It must include $Re\{s\} > -2$. There are two possible ROCs for $\frac{1}{s+3}$:

$$e^{-3t}u(t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} > -3$$

$$-e^{-3t}u(-t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} < -3$$

Thus, the ROC of $\frac{1}{s+3}$ is $Re\{s\} > -3$.

Finally:

$$x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

d) $X(s) = \frac{s+1}{s^2+5s+6}, Re\{s\} < -3$

$$X(s) = \frac{s+1}{s^2+5s+6} = \frac{s}{(s+2)(s+3)} = -\frac{1}{(s+2)} + \frac{2}{(s+3)}, Re\{s\} < -3$$

$\frac{1}{s+2}$: It must include $Re\{s\} < -3$. There are two possible ROCs for $\frac{1}{s+2}$:

$$e^{-2t}u(t) \xleftrightarrow{L} \frac{1}{s+2}, Re\{s\} > -2$$

$$-e^{-2t}u(-t) \xleftrightarrow{L} \frac{1}{s+2}, Re\{s\} < -2$$

Thus, the ROC of $\frac{1}{s+2}$ is $Re\{s\} < -2$

$\frac{1}{s+3}$: It must include $Re\{s\} < -3$. There are two possible ROCs for $\frac{1}{s+3}$:

$$e^{-3t}u(t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} > -3$$

$$-e^{-3t}u(-t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} < -3$$

Thus, the ROC of $\frac{1}{s+3}$ is $Re\{s\} < -3$.

Finally:

$$x(t) = e^{-2t}u(-t) - 2e^{-3t}u(-t)$$

e) $X(s) = \frac{1}{(s+3)(s+4)}, -4 < Re\{s\} < -3$

$$X(s) = \frac{1}{(s+3)(s+4)} = \frac{1}{(s+3)} - \frac{1}{(s+4)}, -4 < Re\{s\} < -3$$

$\frac{1}{s+3}$: It must include $-4 < Re\{s\} < -3$. There are two possible ROCs for $\frac{1}{s+3}$:

$$e^{-3t}u(t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} > -3$$

$$-e^{-3t}u(-t) \xleftrightarrow{L} \frac{1}{s+3}, Re\{s\} < -3$$

Thus, the ROC of $\frac{1}{s+3}$ is $Re\{s\} < -3$

$\frac{1}{s+4}$: It must include $-4 < Re\{s\} < -3$. There are two possible ROCs for $\frac{1}{s+4}$:

$$e^{-4t}u(t) \xleftrightarrow{L} \frac{1}{s+4}, Re\{s\} > -4$$

$$-e^{-4t}u(-t) \xleftrightarrow{L} \frac{1}{s+4}, Re\{s\} < -4$$

Thus, the ROC of $\frac{1}{s+4}$ is $Re\{s\} > -4$.

Finally:

$$x(t) = -e^{-3t}u(-t) - e^{-4t}u(t)$$

PROBLEM 6

For the following rational Laplace Transform, with a certain region of convergence.

$$X(s) = 5\frac{(s+1)(s+2)(s+5)(s-6)}{(s+3)(s+3)s}$$

- Using MATLAB®, show the zero-pole plot. Attach your MATLAB® code to your plot.
- Using MATLAB®, decompose $X(s)$ into partial fractions. Attach your MATLAB® code.
- Provide the expression of the Fourier transform. Then, provide the equation of the magnitude and the phase in terms of the frequency variable.
  Hint: It might be useful to express each resulting complex number in polar form.

- $X(s) = 5\frac{(s+1)(s+2)(s+5)(s-6)}{(s+3)(s+3)s} = \frac{5s^4+10s^3-155s^2-460s-300}{s^3+6s^2+9s}$

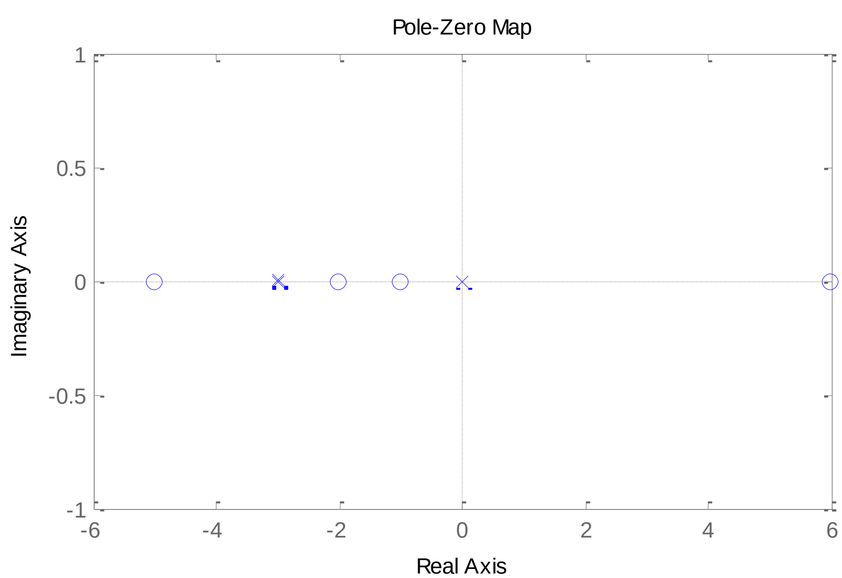
  MATLAB code:

```
clear all; close all; clc;
% X(s) = N(s)/D(s)

pN = [5 10 -155 -460 -300]; % Polynomial for N(s)
pD = [1 6 9 0]; % Polynomial for D(s):

% Getting the zeros:
z = roots(pN);

% Getting the poles:
p = roots(pD);

% Plotting the pole-zero plot (needs the Control System toolbox)
pzmap(p,z); axis([-6 6 -1 1]);

% Partial Fraction expansion:
[r, p, k] = residue(pN,pD);

>> r = -46.6667, 60, -33.3333
>> p = -3, -3, 0
>> k = 5, - 20
```

  Partial Fraction Expansion:

$$\rightarrow X(s) = -\frac{140}{3(s+3)} + \frac{60}{s+3} - \frac{100}{3s} + 5s - 20$$

  Pole-zero plot:



- Fourier Transform: $s = j\omega$:

  $$\rightarrow X(j\omega) = 5\frac{(j\omega+1)(j\omega+2)(j\omega+5)(j\omega-6)}{(j\omega+3)(j\omega+3)j\omega}$$

  Magnitude:

  $$\rightarrow |X(j\omega)| = 5\frac{\left(\sqrt{1+\omega^2}\right)\left(\sqrt{4+\omega^2}\right)\left(\sqrt{25+\omega^2}\right)\left(\sqrt{36+\omega^2}\right)}{(9+\omega^2)\omega}$$

  Angle:

  $$\rightarrow \angle X(j\omega) = \frac{\left(Tan^{-1}(\omega)\right)\left(Tan^{-1}\left({}^{\omega}/_{2}\right)\right)\left(Tan^{-1}\left({}^{\omega}/_{5}\right)\right)\left(Tan^{-1}\left(-{}^{\omega}/_{6}\right)\right)}{2Tan^{-1}\left({}^{\omega}/_{3}\right)+\frac{\pi}{2}}$$