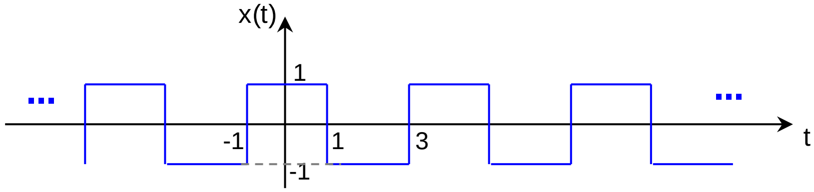
Solutions - Homework # 4

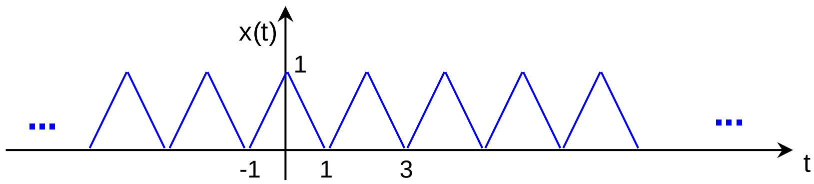
PROBLEM 1

Find the FT representation of the following periodic signals:

a)



b)



c) $x(t) = 2Sin(\pi t) + Cos(2\pi t)$

d) $x(t) = e^{j\omega_0 t}$

e) $x(t) = \left|Sin\left(\frac{\pi}{2}t\right)\right|$

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a) Periodic signal → FS, T = 4, $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$X[k] = \frac{1}{4}\int_{-1}^{3} x(t)e^{-jk\omega_0 t}dt = \frac{1}{4}\int_{-1}^{1} e^{-jk\frac{\pi}{2}t}dt - \frac{1}{4}\int_{1}^{3} e^{-jk\frac{\pi}{2}t}dt$$

$k \neq 0$:

$$X[k] = -\frac{1}{jk2\pi}e^{-jk\frac{\pi}{2}t}\Big|_{-1}^{1} + \frac{1}{jk2\pi}e^{-jk\frac{\pi}{2}t}\Big|_{1}^{3} = -\frac{1}{jk2\pi}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right) + \frac{1}{jk2\pi}\left(e^{-jk\frac{3\pi}{2}} - e^{-jk\frac{\pi}{2}}\right)$$

$$X[k] = \frac{1}{jk2\pi}\left(-e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}} + e^{-jk\frac{3\pi}{2}} - e^{-jk\frac{\pi}{2}}\right), e^{-jk\frac{3\pi}{2}} = e^{-jk2\pi}e^{jk\frac{\pi}{2}} = e^{jk\frac{\pi}{2}}$$

$$X[k] = \frac{1}{jk2\pi}\left(2e^{jk\frac{\pi}{2}} - 2e^{-jk\frac{\pi}{2}}\right) = \frac{2}{\pi k}Sin\left(k\frac{\pi}{2}\right), Note: if\ k\ is\ even, then\ X[k] = 0$$

$k = 0$:

$$X[k] = \frac{1}{4}\int_{-1}^{1} dt - \frac{1}{4}\int_{1}^{3} 1dt = 0$$

Thus:

$$X[k] = \begin{cases} \frac{2}{\pi k}Sin\left(k\frac{\pi}{2}\right), & k\ odd \\ 0, & k\ even \end{cases}$$

Finally:

$$x(t) \overset{FT}{\leftrightarrow} X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k\ odd} \frac{4}{k}Sin\left(k\frac{\pi}{2}\right)\delta\left(\omega - k\frac{\pi}{2}\right)$$

b) Periodic signal → FS, T = 2, $\omega_0 = \frac{2\pi}{2} = \pi$

$$X[k] = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\left(\int_{-1}^{0}(t+1)e^{-jk\pi t}dt + \int_{0}^{1}(1-t)e^{-jk\pi t}dt\right)$$

$$X[k] = \frac{1}{2}\left(\int_{-1}^{0} e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt + \int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt\right)$$

$k = 0$:

$$X[k] = \frac{1}{2}\left(\int_{-1}^{1} dt + \int_{-1}^{0} tdt - \int_{0}^{1} tdt\right) = \frac{1}{2}\left(2 + \frac{t^2}{2}\Big|_{-1}^{0} - \frac{t^2}{2}\Big|_{0}^{1}\right) = \frac{1}{2}\left(2 - \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$$

$k \neq 0$:

$$X[k] = \frac{1}{2}\left(\int_{-1}^{1} e^{-jk\pi t}dt + \int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt\right), \qquad \int_{a}^{b} te^{ct}dt = \frac{1}{c^2}e^{ct}(ct-1)\Big|_a^b$$

$$\int_{-1}^{1} e^{-jk\pi t}dt = -\frac{1}{jk\pi}e^{-jk\pi t}\Big|_{-1}^{1} = -\frac{1}{jk\pi}\left(e^{-jk\pi t} - e^{jk\pi t}\right) = \frac{1}{k\pi}Sin(k\pi) = 0$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = \frac{1}{(-jk\pi)^2}\left(e^{-jk\pi t}(-jk\pi t - 1)\right)\Big|_{-1}^{0} - \frac{1}{(-jk\pi)^2}\left(e^{-jk\pi t}(-jk\pi t - 1)\right)_0^1$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = \frac{1}{(-jk\pi)^2}\left(-1 - e^{jk\pi}(jk\pi - 1)\right) - \frac{1}{(-jk\pi)^2}\left(e^{-jk\pi}(-jk\pi - 1) + 1\right)$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = \frac{1}{(-jk\pi)^2}\left(-1 - e^{jk\pi}(jk\pi - 1) - e^{-jk\pi}(-jk\pi - 1) - 1\right)$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = -\frac{1}{k^2\pi^2}\left(-2 - jk\pi e^{jk\pi} + e^{jk\pi} + jk\pi e^{-jk\pi} + e^{-jk\pi}\right)$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = -\frac{1}{k^2\pi^2}\left(-2 - jk\pi(e^{jk\pi} - e^{-jk\pi}) + e^{jk\pi} + e^{-jk\pi}\right)$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = -\frac{1}{k^2\pi^2}\left(-2 - jk\pi\left(2jSin(k\pi)\right) + 2Cos(k\pi)\right) = -\left(\frac{-2 + 2Cos(k\pi)}{k^2\pi^2}\right)$$

$$\int_{-1}^{0} te^{-jk\pi t}dt - \int_{0}^{1} te^{-jk\pi t}dt = \frac{2}{k^2\pi^2}\left(1 - Cos(k\pi)\right)$$

$$\rightarrow X[k] = \frac{2}{k^2\pi^2}\left(1 - Cos(k\pi)\right) = \begin{cases} \frac{4}{k^2\pi^2}, & k \neq 0, k\ odd \\ 0, & k \neq 0, k\ even \end{cases}$$

Thus:

$$X[k] = \begin{cases} \frac{4}{k^2\pi^2}, & k \neq 0, k\ odd \\ \frac{1}{2}, & k = 0 \\ 0, & otherwise \end{cases}$$

Finally:

$$X(j\omega) = \sum_{k\ odd} \frac{8}{k^2\pi}\delta(\omega - k\pi) + \pi\delta(\omega)$$

c) $x(t) = 2Sin(\pi t) + Cos(2\pi t)$. Periodic signal → FS
$\pi T = 2\pi r, 2\pi T = 2\pi k$
$T = 2r = k \rightarrow r = 1, k = 2 \Rightarrow T = 2, \omega_0 = \pi$
$x(t) = \frac{1}{j}e^{j\pi t} - \frac{1}{j}e^{-j\pi t} + \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t} = X[1]e^{j\pi t} + X[-1]e^{-j\pi t} + X[2]e^{j2\pi t} + X[-2]e^{-j2\pi t}$
Thus:

$$X[k] = \begin{cases} \frac{1}{j}, k = 1 \\ -\frac{1}{j}, k = -1 \\ \frac{1}{2}, k = \pm 2 \\ 0, otherwise \end{cases}$$

Finally:

$$X(j\omega) = \frac{2\pi}{j}\delta(\omega - \pi) - \frac{2\pi}{j}\delta(\omega + \pi) + \pi\delta(\omega - 2\pi) + \pi\delta(\omega + 2\pi)$$

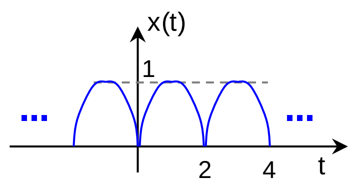
d) $x(t) = e^{j\omega_0 t}$. Periodic signal → FS, ω_0
$x(t) = e^{j\omega_0 t} = X[1]e^{j\omega_0 t}$
Thus:

$$X[k] = \begin{cases} 1, & k = 1 \\ 0, & otherwise \end{cases}$$

Finally:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

e) $x(t) = \left|Sin\left(\frac{\pi}{2}t\right)\right|$. Period Signals → FS. $T = 2 \rightarrow \omega_0 = \pi$



$$X[k] = \frac{1}{2}\int_0^2 x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^2 Sin\left(\frac{\pi}{2}t\right)e^{-jk\pi t}dt$$

$$X[k] = \frac{1}{4j}\int_0^2 e^{j\frac{\pi}{2}t}e^{-jk\pi t}dt - \frac{1}{4j}\int_0^2 e^{-j\frac{\pi}{2}t}e^{-jk\pi t}dt = \frac{1}{4j}\int_0^2 e^{-j\pi t\left(k-\frac{1}{2}\right)}dt - \frac{1}{4j}\int_0^2 e^{-j\pi t\left(k+\frac{1}{2}\right)}dt$$

$$\int_0^2 e^{-j\pi t\left(k-\frac{1}{2}\right)}dt = -\frac{1}{j\pi\left(k-\frac{1}{2}\right)}e^{-j\pi t\left(k-\frac{1}{2}\right)}\Big|_0^2 = -\frac{1}{j\pi\left(k-\frac{1}{2}\right)}\left(e^{-j2\pi\left(k-\frac{1}{2}\right)} - 1\right) = \frac{1}{j\pi\left(k-\frac{1}{2}\right)}(1 - e^{j\pi})$$

$$\int_0^2 e^{-j\pi t\left(k+\frac{1}{2}\right)}dt = -\frac{1}{j\pi\left(k+\frac{1}{2}\right)}e^{-j\pi t\left(k+\frac{1}{2}\right)}\Big|_0^2 = -\frac{1}{j\pi\left(k+\frac{1}{2}\right)}\left(e^{-j2\pi\left(k+\frac{1}{2}\right)} - 1\right) = \frac{1}{j\pi\left(k+\frac{1}{2}\right)}(1 + e^{-j\pi})$$

$$X[k] = \frac{1}{4j}\left(\frac{1}{j\pi\left(k-\frac{1}{2}\right)}(1 - e^{j\pi}) - \frac{1}{j\pi\left(k+\frac{1}{2}\right)}(1 + e^{-j\pi})\right) = -\frac{1}{4\pi}\left(\frac{2}{\left(k-\frac{1}{2}\right)} - \frac{2}{\left(k+\frac{1}{2}\right)}\right)$$

$$X[k] = -\frac{2}{4\pi}\left(\frac{1}{\left(k-\frac{1}{2}\right)} - \frac{1}{\left(k+\frac{1}{2}\right)}\right) = -\frac{2}{4\pi}\left(\frac{1}{k^2 - \frac{1}{4}}\right) = -\frac{1}{2\pi\left(k^2 - \frac{1}{4}\right)}$$

Finally:

$$x(t) \overset{FT}{\leftrightarrow} X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$X(j\omega) = -\sum_{k=-\infty}^{\infty}\frac{1}{\left(k^2 - \frac{1}{4}\right)}\delta(\omega - k\pi)$$

PROBLEM 2

Find the time-domain periodic signal corresponding to the following FT representations:

a) $X(j\omega) = \delta\left(\omega - \frac{\pi}{4}\right) + \delta\left(\omega + \frac{\pi}{4}\right)$

b) $X(j\omega) = 4\pi\delta(\omega - 5\pi) + 2j\pi\delta(\omega - 7\pi) + 4\pi\delta(\omega + 5\pi) - 2j\pi\delta(\omega + 7\pi)$

c) $X(j\omega) = \delta\left(\omega - \frac{\pi}{6}\right) + \delta\left(\omega - \frac{\pi}{8}\right)$

d) $X(j\omega) = \sum_{k=0}^{6} \frac{1}{1+k}\left\{\delta\left(\omega - k\frac{\pi}{3}\right) + \delta\left(\omega + k\frac{\pi}{3}\right)\right\}$

Process: i) Get the FS coefficients $X[k]$, ii) plug them in the equation for $x(t)$.

a) $X(j\omega) = \delta\left(\omega - \frac{\pi}{4}\right) + \delta\left(\omega + \frac{\pi}{4}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \overset{FT}{\leftrightarrow} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$X(j\omega) = 2\pi\left\{X[1]\delta\left(\omega - \frac{\pi}{4}\right) + X[-1]\delta\left(\omega + \frac{\pi}{4}\right)\right\}, \omega_0 = \frac{\pi}{4}$$

$$* \; \omega_0 = \frac{\pi}{8}, \frac{\pi}{16}, \dots \; would\ also\ work, but\ we\ pick\ the\ largest\ to\ have\ the\ smallest\ period$$

Then:

$$X[k] = \begin{cases} \frac{1}{2\pi}, & k = \pm 1 \\ 0, & otherwise \end{cases}$$

Finally:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{1}{2\pi}e^{-j\frac{\pi}{4}t} + \frac{1}{2\pi}e^{j\frac{\pi}{4}t} = \frac{1}{\pi}Cos\left(\frac{\pi}{4}t\right)$$

b) $X(j\omega) = 4\pi\delta(\omega - 5\pi) + 2j\pi\delta(\omega - 7\pi) + 4\pi\delta(\omega + 5\pi) - 2j\pi\delta(\omega + 7\pi)$

$X(j\omega) = 2\pi\{X[5]\delta(\omega - 5\pi) + X[7]\delta(\omega - 7\pi) + X[-5]\delta(\omega + 5\pi) + X[-5]\delta(\omega + 5\pi)\}, \omega_0 = \pi$

Then:

$$X[k] = \begin{cases} 2, & k = \pm 5 \\ j, & k = 7 \\ -j, & k = -7 \\ 0, & otherwise \end{cases}$$

Finally:

$$x(t) = -je^{-j7\pi t} + je^{j7\pi t} + 2e^{j5\pi t} + 2e^{-j5\pi t} = -2Sin(7\pi t) + 4Cos(5\pi t)$$

c) $X(j\omega) = \delta\left(\omega - \frac{\pi}{6}\right) + \delta\left(\omega - \frac{\pi}{8}\right)$

$$\frac{\pi}{6} = k_1\omega_0, \quad \frac{\pi}{8} = k_2\omega_0 \rightarrow \omega_0 = \frac{\pi}{6k_1} = \frac{\pi}{8k_2} \rightarrow 4k_2 = 3k_1 \rightarrow k_2 = 3, k_1 = 4 \Rightarrow \omega_0 = \frac{\pi}{24}$$

$$X(j\omega) = 2\pi\left\{X[4]\delta\left(\omega - 4\frac{\pi}{24}\right) + X[3]\delta\left(\omega - 3\frac{\pi}{24}\right)\right\}$$

Then:

$$X[k] = \begin{cases} \frac{1}{2\pi}, & k = 3, 4 \\ 0, & otherwise \end{cases}$$

Finally:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{1}{2\pi}e^{j\frac{\pi}{6}t} + \frac{1}{2\pi}e^{j\frac{\pi}{8}t}$$

d) $X(j\omega) = \sum_{k=0}^{6} \frac{1}{1+k}\left\{\delta\left(\omega - k\frac{\pi}{3}\right) + \delta\left(\omega + k\frac{\pi}{3}\right)\right\}$

$$X(j\omega) = \sum_{k=0}^{6} \frac{1}{1+k}\left\{\delta\left(\omega - k\frac{\pi}{3}\right) + \delta\left(\omega + k\frac{\pi}{3}\right)\right\}, \omega_0 = \frac{\pi}{3}$$

$$X(j\omega) = 2\delta(\omega) + \frac{1}{2}\left\{\delta\left(\omega - \frac{\pi}{3}\right) + \delta\left(\omega + \frac{\pi}{3}\right)\right\} + \frac{1}{3}\left\{\delta\left(\omega - 2\frac{\pi}{3}\right) + \delta\left(\omega + 2\frac{\pi}{3}\right)\right\} + \frac{1}{4}\{\delta(\omega - \pi) + \delta(\omega + \pi)\}$$
$$+\frac{1}{5}\left\{\delta\left(\omega - 4\frac{\pi}{3}\right) + \delta\left(\omega + 4\frac{\pi}{3}\right)\right\} + \frac{1}{6}\left\{\delta\left(\omega - 5\frac{\pi}{3}\right) + \delta\left(\omega + 5\frac{\pi}{3}\right)\right\} + \frac{1}{7}\{\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \overset{FT}{\leftrightarrow} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

Then:

$$X[k] = \begin{cases} \frac{1}{\pi}, k = 0 & \frac{1}{10\pi}, k = \pm 4 \\ \frac{1}{4\pi}, k = \pm 1 & \frac{1}{12\pi}, k = \pm 5 \\ \frac{1}{6\pi}, k = \pm 2 & \frac{1}{14\pi}, k = \pm 6 \\ \frac{1}{8\pi}, k = \pm 3 & \end{cases} \equiv X[k] = \begin{cases} \frac{1}{\pi}, & k = 0 \\ \frac{1}{2\pi(1+|k|)}, & k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \end{cases}$$

Finally:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{1}{\pi} + \sum_{k=1}^{6} \frac{1}{2\pi(1+|k|)} e^{jk\frac{\pi}{3}t} + \sum_{k=-6}^{-1} \frac{1}{2\pi(1+|k|)} e^{jk\frac{\pi}{3}t}$$

$$x(t) = \frac{1}{\pi} + \sum_{k=1}^{6} \frac{1}{2\pi(1+|k|)} e^{jk\frac{\pi}{3}t} + \sum_{k=1}^{6} \frac{1}{2\pi(1+|k|)} e^{-jk\frac{\pi}{3}t} = \frac{1}{\pi} + \sum_{k=1}^{6} \frac{1}{2\pi(1+|k|)}\left\{e^{jk\frac{\pi}{3}t} + e^{-jk\frac{\pi}{3}t}\right\}$$

$$x(t) = \frac{1}{\pi} + \sum_{k=1}^{6} \frac{1}{\pi(1+|k|)} Cos\left(k\frac{\pi}{3}t\right) = \sum_{k=0}^{6} \frac{1}{\pi(1+|k|)} Cos\left(k\frac{\pi}{3}t\right)$$

PROBLEM 3

The DTFT formula for periodic signals is usually specified as:

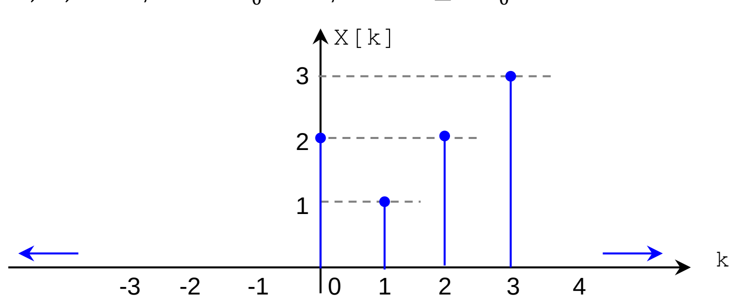
$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n} \overset{DTFT}{\longleftrightarrow} X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_0)$$

Since $X[k]$ is N-periodic, a simpler way to specify the DTFT is to specify it only over one period (2π):

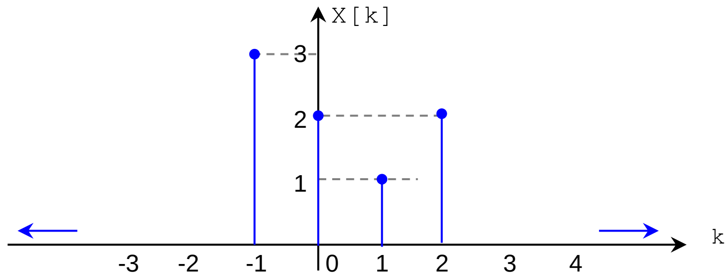
$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n} \overset{DTFT}{\longleftrightarrow} X(e^{j\Omega}) = 2\pi \sum_{k=0}^{N-1} X[k]\delta(\Omega - k\Omega_0)$$

Once you specify over one 2π-period, all you need is to generate infinite periodic replicas.

a) Given the following DTFS coefficients $X[k]$ for one period ($k = 0, \ldots, N-1$), sketch the DTFT representation for one 2π-period. Then, generate the DTFT representation for all periods by generating infinite replicas on both sides. N = 4.
Notice that since $k = 0, \ldots, N-1$, and $N\Omega_0 = 2\pi$, then $0 \le k\Omega_0 < 2\pi$

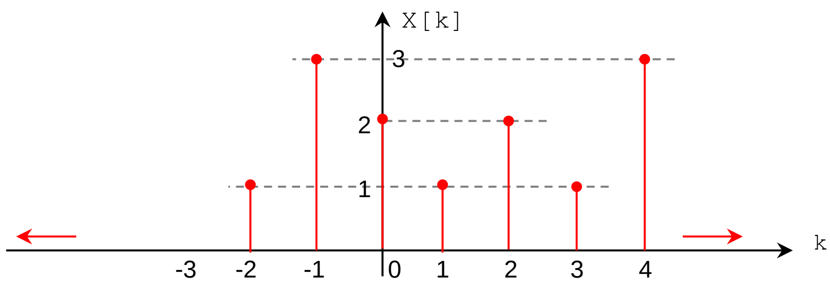


b) Now, you are given the DTFS coefficients $X[k]$ for one period ($k = -1, \dots, N-2$). Sketch the DTFT representation for one 2π-period. Then, generate the DTFT representation for all periods by generating infinite replicas on both sides.
Notice that since $k = -1, \dots, N-2$, and $N\Omega_0 = 2\pi$, then $-\Omega_0 \le k\Omega_0 < 3\Omega_0$ (a 2π range)



c) The two DTFT representations of (a) and (b) must be the same. Actually, you could pick any N-period in $X[k]$ and get the same DTFT. For example, sketch $X[k]$ for one period ($k = -(N-2), \dots, 1$), and show that the DTFT representation is the same as those of (a) and (b).
Notice that since $k = -(N-2), \dots, 1$, and $N\Omega_0 = 2\pi$, then $-2\Omega_0 \le k\Omega_0 < 2\Omega_0$ (a 2π range)

d) Given the following DTFS coefficients $X[k]$ for one N-period ($k = -2, \dots, 4$), sketch $X[k]$ for all periods. Then, redefine one period of $X[k]$ for the following periods: $k = 0, \dots, N-1$ and $k = -\left\lfloor \frac{N}{2} \right\rfloor, \dots, \left\lceil \frac{N}{2} \right\rceil - 1$.



e) When referring to the DTFT over one period, it is customary to pick $0 \le k\Omega_0 < 2\pi$, or $-\pi \le k\Omega_0 < \pi$.
For N = 7 and 8, demonstrate that:
 i. If $k = 0, \dots, N-1$, and $N\Omega_0 = 2\pi$, then $0 \le k\Omega_0 < 2\pi$.
 ii. If $k = -\left\lfloor \frac{N}{2} \right\rfloor, \dots, \left\lceil \frac{N}{2} \right\rceil - 1$, and $N\Omega_0 = 2\pi$, then $-\pi \le k\Omega_0 < \pi$.

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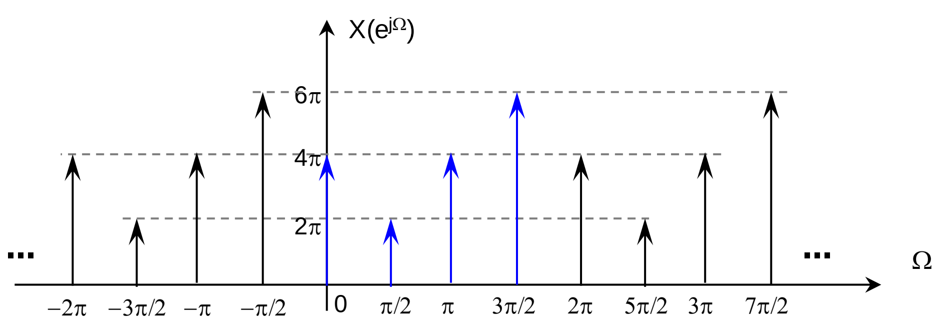
a) $N = 4 \rightarrow \Omega_0 = \frac{\pi}{2}$. For one period:

$$X(e^{j\Omega}) = 2\pi \sum_{k=0}^{3} X[k]\delta\left(\Omega - k\frac{\pi}{2}\right) = 2\pi\left(2\delta(\Omega) + \delta\left(\Omega - \frac{\pi}{2}\right) + 2\delta\left(\Omega - 2\frac{\pi}{2}\right) + 3\delta\left(\Omega - 3\frac{\pi}{2}\right)\right)$$

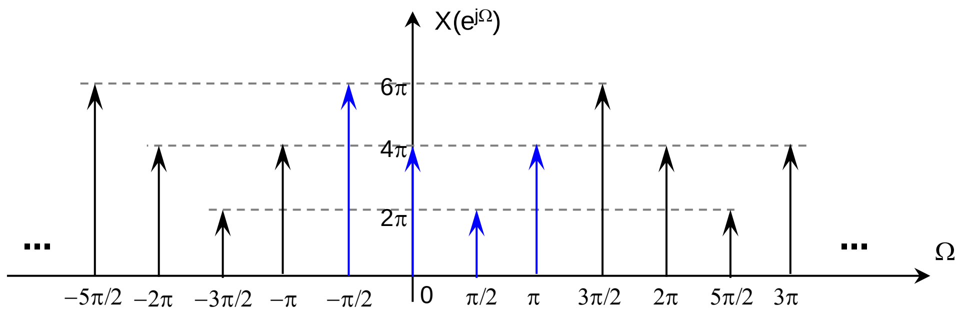
$$0 \le k \le 3 \rightarrow 0 \le k\frac{\pi}{2} \le 3\frac{\pi}{2} \equiv 0 \le k\frac{\pi}{2} < 2\pi$$



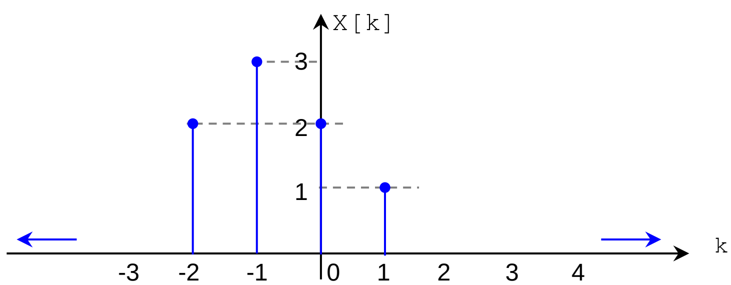
b)

$$X(e^{j\Omega}) = 2\pi \sum_{k=-1}^{2} X[k]\delta\left(\Omega - k\frac{\pi}{2}\right) = 2\pi\left(3\delta\left(\Omega + \frac{\pi}{2}\right) + 2\delta(\Omega) + \delta\left(\Omega - \frac{\pi}{2}\right) + 2\delta\left(\Omega - 2\frac{\pi}{2}\right)\right)$$

$$-1 \le k \le 2 \rightarrow -\frac{\pi}{2} \le k\frac{\pi}{2} \le 2\pi \equiv -\frac{\pi}{2} \le k\frac{\pi}{2} < 3\frac{\pi}{2}$$

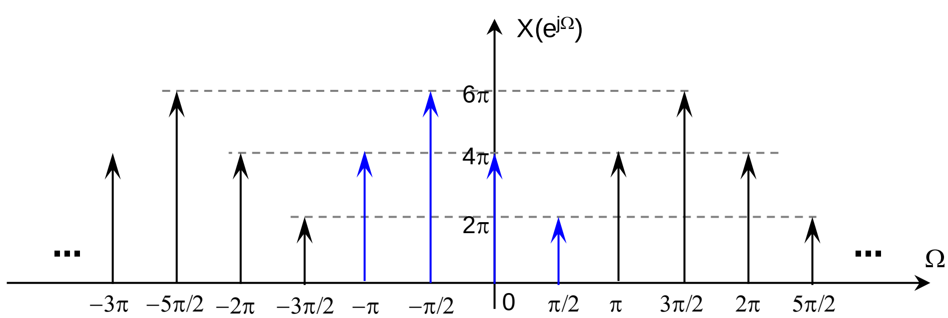


c)

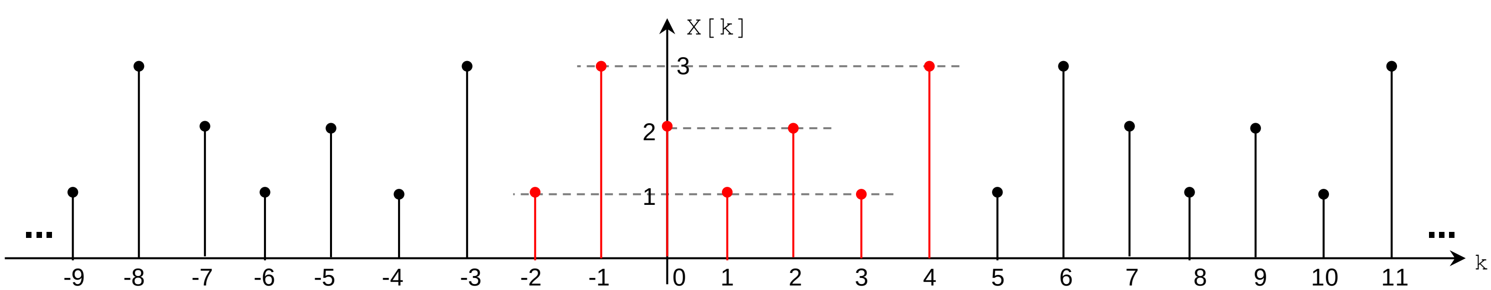


$$X(e^{j\Omega}) = 2\pi \sum_{k=-2}^{1} X[k]\delta\left(\Omega - k\frac{\pi}{2}\right) = 2\pi\left(2\delta\left(\Omega + 2\frac{\pi}{2}\right) + 3\delta\left(\Omega + \frac{\pi}{2}\right) + 2\delta(\Omega) + \delta\left(\Omega - \frac{\pi}{2}\right)\right)$$

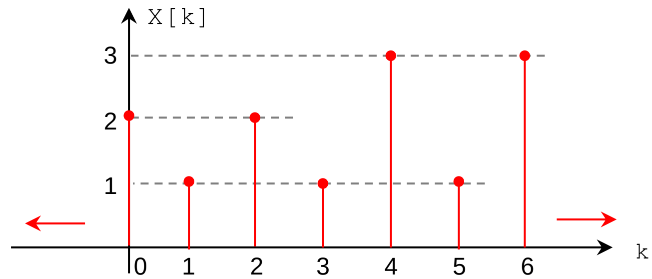
$$-2 \le k \le 1 \rightarrow -\pi \le k\frac{\pi}{2} \le \frac{\pi}{2} \equiv -\pi \le k\frac{\pi}{2} < \pi$$



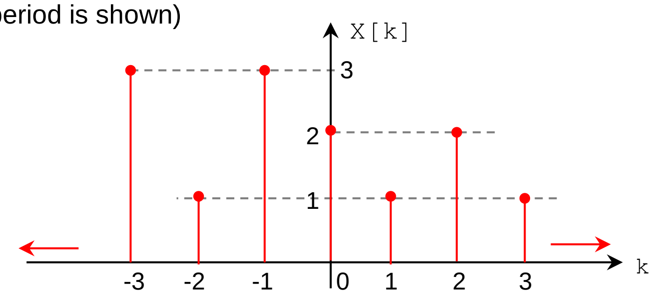
d) Period: $N = 7$: $k = -2, \ldots, 4$: (with the infinite copies to both sides)



$k = 0, \dots, 6$: (only one period is shown)



$k = -3, \dots, 3$: (only one period is shown)



e) DTFT over one period:

i. $k = 0, \dots, N-1,\ N\Omega_0 = 2\pi$:

$$0 \le k\Omega_0 \le (N-1)\Omega_0$$

$$N = 7 \to \Omega_0 = \frac{2\pi}{7} \to 0 \le k\Omega_0 \le 6\frac{2\pi}{7} < 2\pi \Rightarrow 0 \le k\Omega_0 < 2\pi$$

$$N = 8 \to \Omega_0 = \frac{\pi}{4} \to 0 \le k\Omega_0 \le 7\frac{\pi}{4} < 2\pi \Rightarrow 0 \le k\Omega_0 < 2\pi$$

ii. $k = -\left\lfloor \frac{N}{2} \right\rfloor, \dots, \left\lceil \frac{N}{2} \right\rceil - 1,\ N\Omega_0 = 2\pi$:

$$-\left\lfloor \frac{N}{2} \right\rfloor \Omega_0 \le k\Omega_0 \le \left(\left\lceil \frac{N}{2} \right\rceil - 1\right)\Omega_0$$

$$N = 7 \to \Omega_0 = \frac{2\pi}{7} \to -3\frac{2\pi}{7} \le k\Omega_0 \le 3\frac{2\pi}{7} \Rightarrow -\pi \le k\Omega_0 < \pi$$

$$N = 8 \to \Omega_0 = \frac{\pi}{4} \to -4\frac{\pi}{4} \le k\Omega_0 \le 3\frac{\pi}{4} \Rightarrow -\pi \le k\Omega_0 < \pi$$

PROBLEM 4

Find the DTFT representation of the following periodic signals:

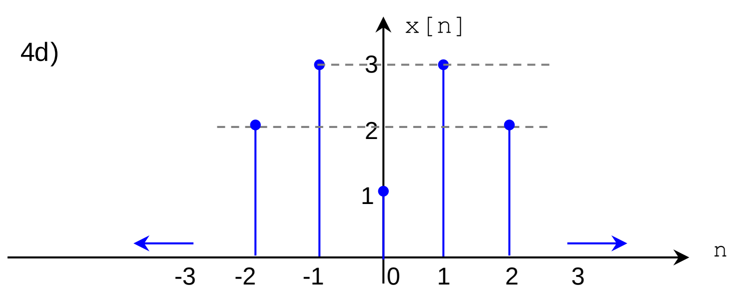
a) $x[n] = Sin\left(\frac{9\pi}{16}n\right)$

b) $x[n] = e^{j\frac{\pi}{3}n + \frac{\pi}{2}}$

c) $x[n] = jcos\left(\frac{5\pi}{7}n\right) + sin\left(\frac{5\pi}{7}n\right)$

d) see figure →

4d)



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a) $x[n] = Sin\left(\frac{9\pi}{16}n\right) \to N = \frac{2\pi}{\frac{9\pi}{16}}m = \frac{32}{9}m \to N = 32, m = 9 \to \Omega_0 = \frac{\pi}{16}$

$$x[n] = \frac{1}{2j}e^{j\frac{9\pi}{16}n} - \frac{1}{2j}e^{-j\frac{9\pi}{16}n} = X[9]e^{j\frac{9\pi}{16}n} + X[-9]e^{-j\frac{9\pi}{16}n}$$

Then: $X[k] = \begin{cases} \frac{1}{2j}, & k = 9 \\ -\frac{1}{2j}, & k = -9 \\ 0, & otherwise\ in\ the\ period: k = -9, \dots 22 \end{cases}$

Finally:

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\frac{\pi}{16}\right)$$

Over one period:

$$X(e^{j\Omega}) = \frac{\pi}{j}\left(\delta\left(\Omega - 9\frac{\pi}{16}\right) - \delta\left(\Omega + 9\frac{\pi}{16}\right)\right), -9 \le k \le 22$$

b) $x[n] = e^{j\frac{\pi}{3}n+\frac{\pi}{2}} = e^{\frac{\pi}{2}}e^{j\frac{\pi}{3}n} \to N = \frac{2\pi}{\frac{\pi}{3}}m = 6m \to N = 6, m = 1 \to \Omega_0 = \frac{\pi}{3}$

$x[n] = e^{\frac{\pi}{2}}e^{j\frac{\pi}{3}n} = X[1]e^{j\frac{\pi}{3}n}$

Then:

$$X[k] = \begin{cases} e^{\frac{\pi}{2}}, & k = 1 \\ 0, & k = 0,2,3,4,5 \end{cases}, 0 \le k \le 5$$

Finally:

$$X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\frac{\pi}{3}\right)$$

Over one period:

$$X\left(e^{j\Omega}\right) = 2\pi e^{\frac{\pi}{2}}\delta\left(\Omega - \frac{\pi}{3}\right), 0 \le k \le 5$$

c) $x[n] = jcos\left(\frac{5\pi}{7}n\right) + sin\left(\frac{5\pi}{7}n\right) \to N = \frac{2\pi}{\frac{5\pi}{7}}m = \frac{14}{5}m \to N = 14, m = 5, \Omega_0 = \frac{\pi}{7}$

$x[n] = j\left\{cos\left(\frac{5\pi}{7}n\right) - jsin\left(\frac{5\pi}{7}n\right)\right\} = je^{-j\frac{5\pi}{7}n} = X[-5]e^{j\frac{\pi}{3}n}$

Then:

$$X[k] = \begin{cases} j, & k = -5 \\ 0, & otherwise\ in\ the\ period: k = -5, \ldots 8 \end{cases}$$

Finally:

$$X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\frac{\pi}{7}\right)$$

Over one period:

$$X\left(e^{j\Omega}\right) = 2\pi j\delta\left(\Omega + \frac{5\pi}{7}\right), -5 \le k \le 8$$

d) We first determine the DTFS: $N = 5, \Omega_0 = \frac{2\pi}{5}$

$X[k] = \frac{1}{N}\sum_{N} x[n]e^{-jk\Omega_0 n} = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk\Omega_0 n} = \frac{1}{5}\left(2e^{jk\frac{4\pi}{5}} + 3e^{jk\frac{2\pi}{5}} + 1 + 3e^{-jk\frac{2\pi}{5}} + 2e^{-jk\frac{4\pi}{5}}\right)$

$X[k] = \frac{4}{5}Cos\left(\frac{4\pi}{5}k\right) + \frac{6}{5}Cos\left(\frac{2\pi}{5}k\right) + 1, \quad -2 \le k \le 2$

Finally:

$$X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\frac{2\pi}{5}\right)$$

Over one period:

$$X\left(e^{j\Omega}\right) = \frac{2\pi}{5}\sum_{k=-2}^{2}\left\{4Cos\left(\frac{4\pi}{5}k\right) + 6Cos\left(\frac{2\pi}{5}k\right) + 1\right\}\delta\left(\Omega - k\frac{2\pi}{5}\right), -2 \le k \le 2$$

PROBLEM 5

Given the following periodic signal:

$$x(t) = Sin\left(\frac{\pi}{4}t\right) + Sin\left(\frac{\pi}{8}t\right)$$

a) Determine the FT of this periodic signal (Hint: use the inspection method to get the FS). Sketch it.
b) If we sample $x(t)$ with T_S as the sampling period, get the FT of the impulse-sampled signal $x_\delta(t)$.
c) Sketch the FT of the sampled signal for T_S = 2, 5, 8 seconds.
d) What is the minimum sampling period T_S that avoids aliasing in the frequency domain?

---

a) $x(t) = Sin\left(\frac{\pi}{4}t\right) + Sin\left(\frac{\pi}{8}t\right) \rightarrow T = \frac{2\pi}{\frac{\pi}{4}}k = 8k, T = \frac{2\pi}{\frac{\pi}{8}}r = 16r \rightarrow k = 2r \rightarrow k = 2, r = 1 \rightarrow T = 16, \omega_0 = \frac{\pi}{8}$

Fourier Series:

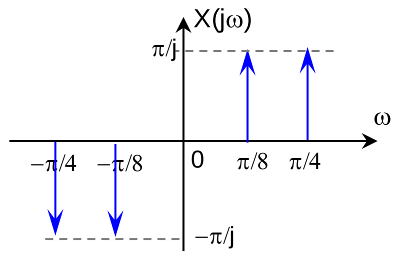
$$x(t) = \frac{1}{2j}e^{j\frac{\pi}{4}t} - \frac{1}{2j}e^{-j\frac{\pi}{4}t} + \frac{1}{2j}e^{j\frac{\pi}{8}t} - \frac{1}{2j}e^{-j\frac{\pi}{8}t} = X[2]e^{j\frac{\pi}{4}t} + X[-2]e^{j\frac{\pi}{4}t} + X[1]e^{j\frac{\pi}{8}t} + X[-1]e^{-j\frac{\pi}{8}t}$$

Then:

$$X[k] = \begin{cases} \frac{1}{2j}, & k = 1,2 \\ -\frac{1}{2j}, & k = -1,-2 \\ 0, & otherwise \end{cases}$$

Finally:

$$X(j\omega) = \frac{\pi}{j}\delta\left(\omega - \frac{\pi}{8}\right) + \frac{\pi}{j}\delta\left(\omega - \frac{\pi}{4}\right) - \frac{\pi}{j}\delta\left(\omega + \frac{\pi}{8}\right) - \frac{\pi}{j}\delta\left(\omega + \frac{\pi}{4}\right)$$

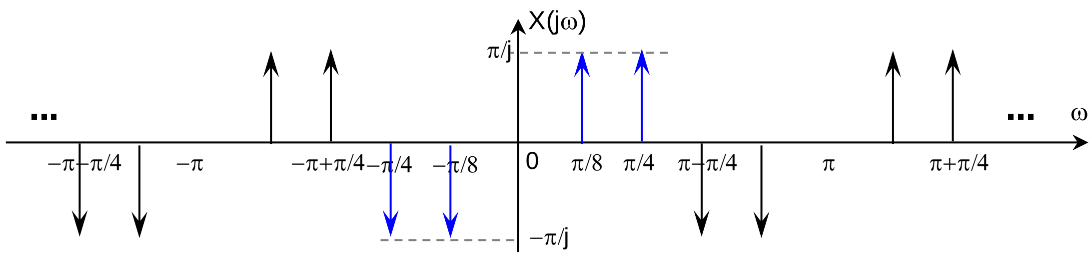


b) $T_S,\ \omega_S = \frac{2\pi}{T_S}$

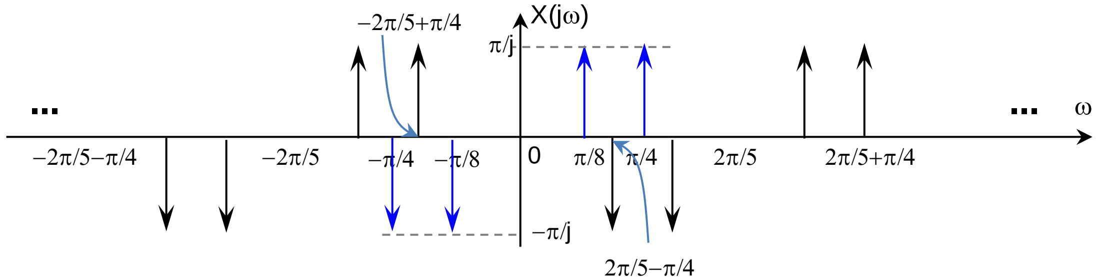
$$X(j\omega) = \frac{1}{T_S}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_S)\big)$$

$$X(j\omega) = \frac{\pi}{jT_S}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{8} - k\omega_S\right) + \delta\left(\omega - \frac{\pi}{4} - k\omega_S\right) - \delta\left(\omega + \frac{\pi}{8} - k\omega_S\right) - \delta\left(\omega + \frac{\pi}{4} - k\omega_S\right)$$

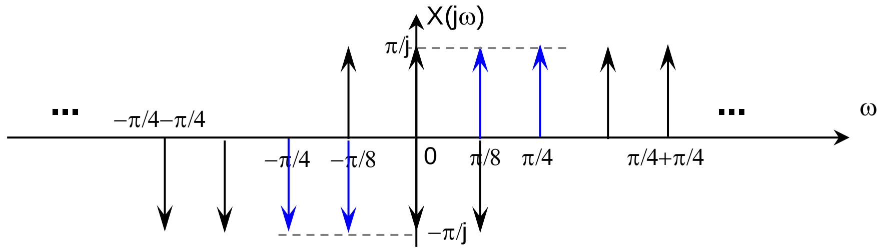
c) $T_S = 2,\ \omega_S = \pi: \omega_S \geq 2\omega_m\ (no\ aliasing)$



$T_S = 5,\ \omega_S = \frac{2\pi}{5}: \omega_S < 2\omega_m (aliasing)$



$T_S = 8,\ \omega_S = \frac{\pi}{4}: \omega_S < 2\omega_m (aliasing)$



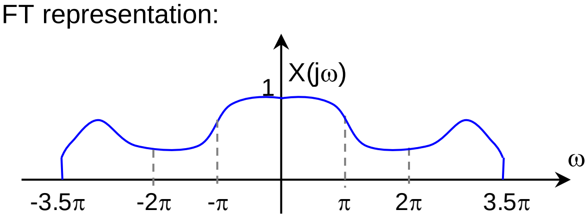
d) From (a), we see that $\omega_m = \frac{\pi}{4}$.

$\rightarrow \omega_S \geq 2\omega_m \ \rightarrow \omega_S > 2\frac{\pi}{4} \rightarrow \frac{2\pi}{T_S} \geq \frac{\pi}{2} \rightarrow T_S < 4\ secs$

The min. sampling period that prevents aliasing is 3.9 seconds. Note that $T_S = 4$ results in aliasing.

PROBLEM 6

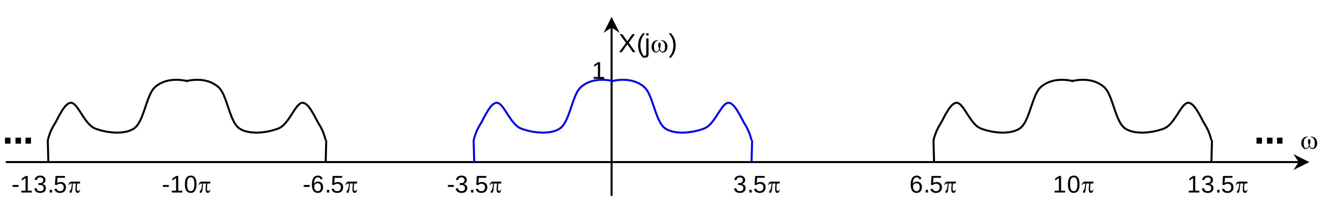
A signal has the following FT representation:



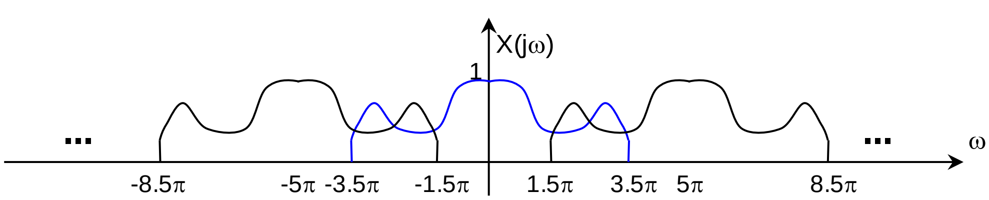
a) The signal is then sampled by T_S seconds. Sketch the FT representation of the sampled signal for the following sampling periods: $T_S = 0.2, 0.4, 0.6$ seconds.
b) What is the condition on the sampling period T_S so that the sampling does not introduce aliasing?

---

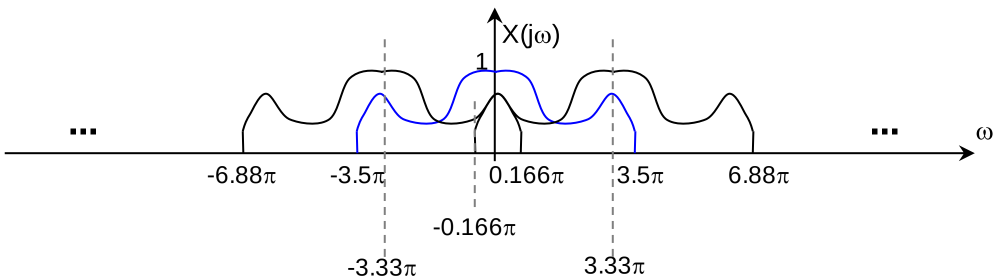
a) $T_S = 0.2,\ \omega_S = 10\pi: \omega_S \geq 2\omega_m\ (no\ aliasing)$



$T_S = 0.4,\ \omega_S = 5\pi: \omega_S < 2\omega_m\ (aliasing)$



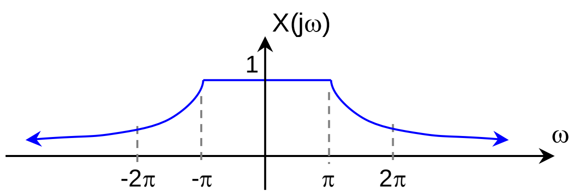
$T_S = 0.6,\ \omega_S = \frac{10}{3}\pi: \omega_S < 2\omega_m\ (aliasing)$



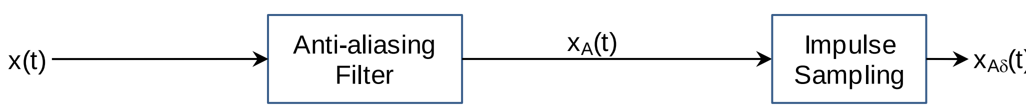
b) $\omega_S \geq 2\omega_m \rightarrow \omega_S \geq 7\pi \rightarrow \frac{2\pi}{T_S} \geq 7\pi \rightarrow T_S \leq 2/7\ secs$

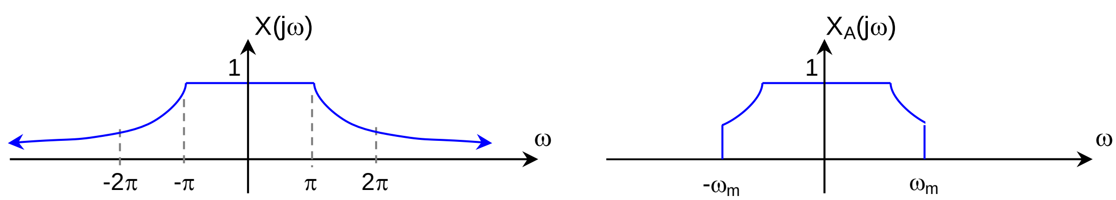
PROBLEM 7

A signal has the following FT representation:



a) Sampling this signal will always introduce aliasing. For this reason, the signal is first passed through a system called anti-aliasing filter, which bounds the FT of the signal (as shown in the figure below). For T_S = 0.5, 0.25, 0.125 seconds, what is the minimum width (in rads/s) of the anti-aliasing filter so that no aliasing is introduced to the signal $x_A(t)$?





b) Sketch the FT of the anti-aliasing filter (it should go from $-\omega_m\ to\ \omega_m$). Also, provide the time-domain impulse response of the anti-aliasing filter.

c) Now, we want to recover $x_A(t)$ after the sampling process. We need an ideal reconstruction filter. Sketch the operation this filter applies on the FT of the sampled signal $x_{A\delta}(t)$. Also provide the time-domain impulse response of this ideal reconstruction filter (use a generic ω_s with $\omega_S = {}^{2\pi}/_{T_S}$).

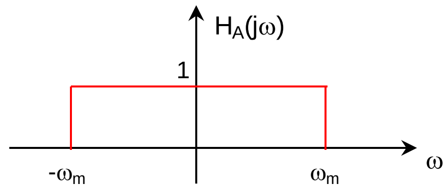
---

a) $T_S = 0.5,\ \omega_S = 4\pi: \omega_S \geq 2\omega_m \rightarrow \omega_m \leq \frac{\omega_S}{2} \rightarrow \omega_m \leq 2\pi$

$T_S = 0.25,\ \omega_S = 8\pi: \omega_S \geq 2\omega_m \rightarrow \omega_m \leq \frac{\omega_S}{2} \rightarrow \omega_m \leq 4\pi$

$T_S = 0.125,\ \omega_S = 16\pi: \omega_S \geq 2\omega_m \rightarrow \omega_m \leq \frac{\omega_S}{2} \rightarrow \omega_m \leq 8\pi$

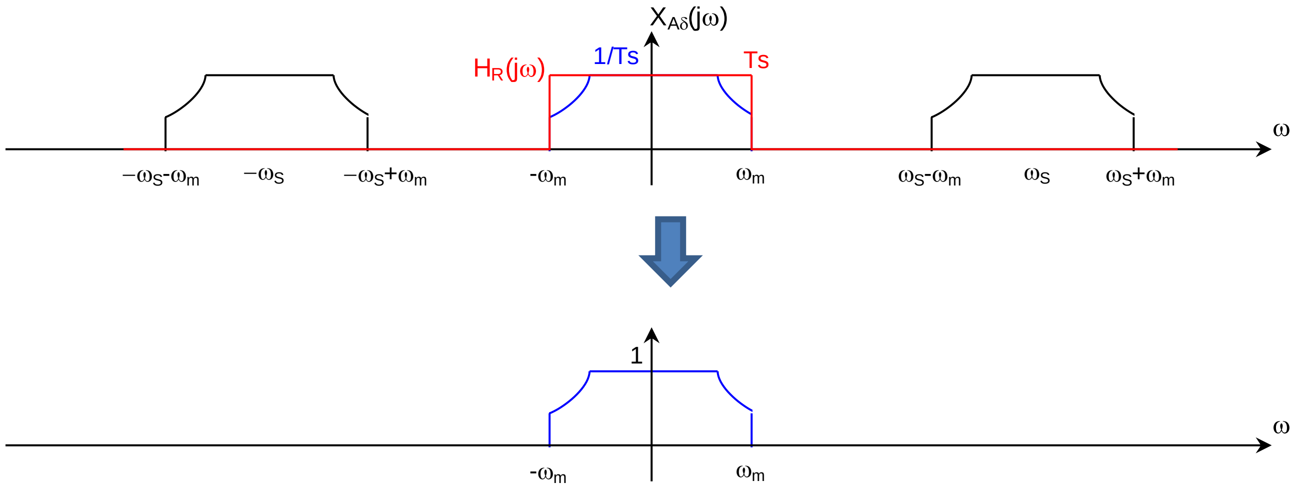
b)



$$h_A(t) = \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} e^{j\omega t} d\omega = \frac{1}{2\pi j t}\left(e^{j\omega_m t} - e^{j-\omega_m t}\right) = \frac{1}{\pi t} Sin(\omega_m t), t \neq 0$$

$$h_A(t) = \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} d\omega = \frac{\omega_m}{\pi}, t = 0$$

c) $\omega_S \geq 2\omega_m$:



$$h_r(t) = \frac{T_S}{2\pi}\int_{-\omega_m}^{\omega_m} e^{j\omega t} d\omega = \frac{T_S}{\pi t} Sin(\omega_m t), t \neq 0$$

$$h_r(t) = \frac{T_S}{2\pi}\int_{-\omega_m}^{\omega_m} d\omega = \frac{T_S \omega_m}{\pi}, t = 0$$