Solutions - Homework # 3

PROBLEM 1

One period of the DTFS coefficients is given by:

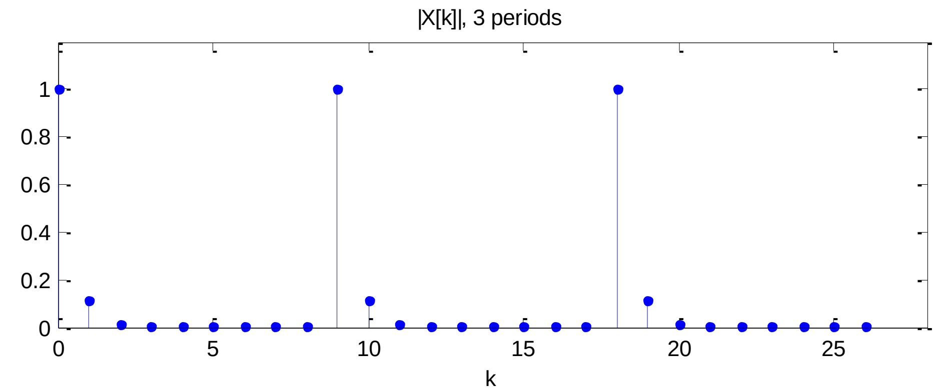
$X[k] = (1/3)^{2k}, \ 0 \le k \le 8.$

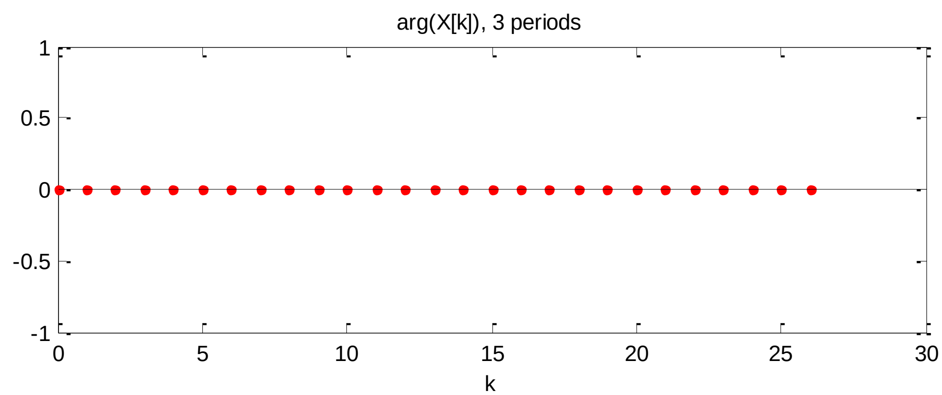
a) What is the fundamental period 'N' of the time-domain signal x[n]?
b) Using MATLAB®, plot X[k] for three periods. Plot the magnitude and the phase spectra.
c) Find the time-domain signal x[n] (provide x[n] as a function of 'n'). Plot x[n] for three periods.

---

a) The period of $x[n]$ is the same as that of $X[k]$. N = 9.
b) $X[k] = (1/3)^{2k}, \ 0 \le k \le 8.$
Note that this equation only works for one period. We have to generate infinite replicas on both sides.

```
clear all; close all; clc
k = 0:8;
X = (1/3).^(2*k); % The function as given only works from 0 to 8
X_3p = [X X X]; % Here, we generate replicas

k_3p = 0:26; n = 0:26; % 3 periods
x = 8./(9 - exp(1i*2*n*pi/9)); % This function is periodic (N=9), so it
                                % works for all 'n'
figure;
subplot (2,1,1), stem(k_3p, abs(X_3p),'.b'); axis ([0 28 0 1.2]);
set(gca, 'Fontsize',8); xlabel('k'); title ('|X[k]|, 3 periods');
subplot (2,1,2), stem(k_3p, angle(X_3p),'.r');
set(gca, 'Fontsize',8); xlabel('k'); title ('arg(X[k]), 3 periods');

figure;
subplot (2,1,1), stem(n,real(x),'.b');
set(gca, 'Fontsize',8); xlabel('n'); title ('Re(x[n]), 3 periods');
subplot (2,1,2), stem(n,imag(x),'.r');
set(gca, 'Fontsize',8); xlabel('n'); title ('Im(x[n]), 3 periods');
```

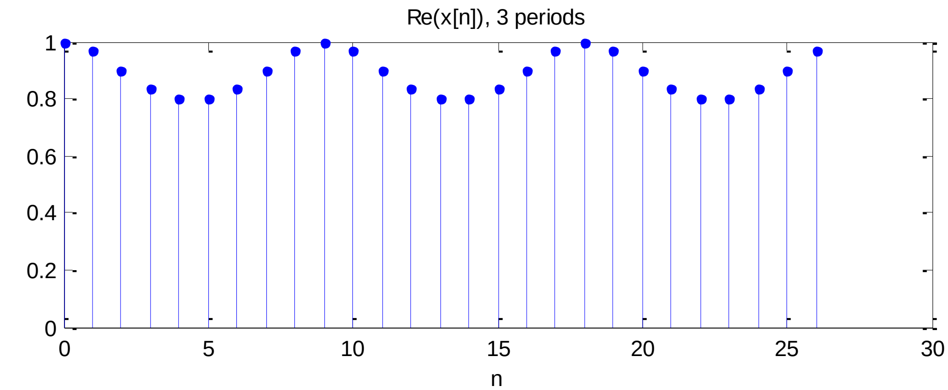


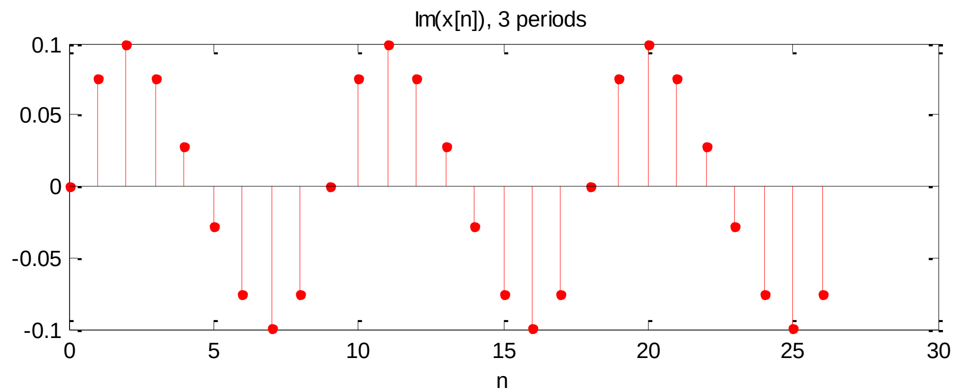


c) $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$

$$x[n] = \sum_{k=0}^{N-1}\left(\frac{1}{3}\right)^{2k} e^{jk\Omega_0 n} = \sum_{k=0}^{8}\left(\frac{1}{9}e^{j\Omega_0 n}\right)^{k} = \frac{1-\frac{1}{9}e^{j\Omega_0 n9}}{1-\frac{1}{9}e^{j\Omega_0 n}} = \frac{1-\frac{1}{9}e^{j2\pi}}{1-\frac{1}{9}e^{j\frac{2\pi}{9}n}} = \frac{1-\frac{1}{9}}{1-\frac{1}{9}e^{j\frac{2\pi}{9}n}}$$

$$x[n] = \frac{8}{9-e^{j\frac{2\pi}{9}n}}, \qquad periodic\ with\ N = 9, e.g., 0 \le n \le 8$$





PROBLEM 2

Identify the appropriate Fourier representation (FT, DTFT, FS, DTFS) for each of the following signals. If the signals are periodic, provide the fundamental period and the fundamental angular frequency

a) x[n] = cos((6π/13)n + π/3)	e) x(t) = sin((π/5)t)
b) x[n] = exp(j(π/4)n)	f) x(t) = cos((π/3)t + π/5)
c) x(t) = cos(t/6)	g) x[n] = δ[n+2] + δ[n-4]
d) x(t) = e^{1-t}u(-t + 2)	h) x[n] = $(3/8)^n$u[n-3]

Once you identified the appropriate Fourier representation, use the defining equation to obtain the DTFS coefficients, the FS coefficients, the DTFT, or the FT.

a) $x[n] = Cos\left(\frac{6\pi}{13}n + \frac{\pi}{3}\right) \rightarrow \frac{6\pi}{13}N = 2\pi m \rightarrow N = \frac{13}{3}m \rightarrow N = 13,\ m = 3$

Signal is periodic with $N = 13, \Omega_0 = \frac{2\pi}{13} \rightarrow$ DTFS

Since the signal is a cosine, we can use the inspection method:

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n} = \frac{e^{j\left(\frac{6\pi}{13}n+\frac{\pi}{3}\right)} + e^{-j\left(\frac{6\pi}{13}n+\frac{\pi}{3}\right)}}{2} = \frac{e^{j\frac{\pi}{3}}}{2}e^{j\left(\frac{6\pi}{13}n\right)} + \frac{e^{-j\frac{\pi}{3}}}{2}e^{-j\left(\frac{6\pi}{13}n\right)}$$

$$x[n] = \frac{e^{j\frac{\pi}{3}}}{2}e^{j(3\Omega_0 n)} + \frac{e^{-j\frac{\pi}{3}}}{2}e^{-j(3\Omega_0 n)} = X[3]e^{j(3\Omega_0 n)} + X[-3]e^{-j(3\Omega_0 n)}$$

Thus, we have the DTFS coefficients $X[k]$ for one period ($k = -3\ to\ 9$):

$$X[k] = \begin{cases} \frac{e^{-j\frac{\pi}{3}}}{2}, & k = -3 \\ \frac{e^{j\frac{\pi}{3}}}{2}, & k = 3 \\ 0, & k = -2,-1,0,1,4,5,6,7,8,9 \end{cases}$$

b) $x[n] = e^{j\frac{\pi}{4}n} \rightarrow \frac{\pi}{4}N = 2\pi m \rightarrow N = 8m \rightarrow N = 8,\ m = 1$
Signal is periodic with $N = 8, \Omega_0 = \frac{\pi}{4} \rightarrow$ DTFS
Since the signal is a complex exponential, we can use the inspection method:
$x[n] = e^{j\frac{\pi}{4}n} = X[1]e^{j\Omega_0 n}$
Thus, we have the DTFS coefficients $X[k]$ for one period ($k = 0\ to\ 7$):

$$X[k] = \begin{cases} 1, & k = 1 \\ 0, & k = 0,2,3,4,5,6,7 \end{cases}$$

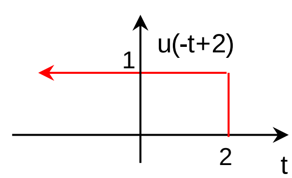
c) $x(t) = Cos\left(\frac{t}{6}\right) \rightarrow \frac{1}{6}T = 2\pi m \rightarrow T = 12\pi m \rightarrow T = 12\pi,\ m = 1$
Signal is periodic with $T = 12\pi, \omega_0 = \frac{1}{6} \rightarrow$ FS
Since the signal is a cosine, we can use the inspection method:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{e^{j\left(\frac{t}{6}\right)} + e^{-j\left(\frac{t}{6}\right)}}{2} = \frac{1}{2}e^{j\left(\frac{t}{6}\right)} + \frac{1}{2}e^{-j\left(\frac{t}{6}\right)} = X[1]e^{j\omega_0 t} + X[-1]e^{-j\omega_0 t}$$

Thus, we have the FS coefficients $X[k]$ for all k: $X[k] = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & otherwise \end{cases}$

d) $x(t) = e^{1-t}u(2-t)$
Non-periodic $\rightarrow$ FT

$$X(j\omega) = \int_{-\infty}^{2} e^{1-t}e^{-j\omega t}dt = e\int_{-\infty}^{2} e^{-t(1+j\omega)}dt = \frac{-e}{1+j\omega}e^{-t(1+j\omega)}\Big|_{-\infty}^{2}$$



The integral diverges, thus the FT representation does not exist.

e) $x(t) = Sin\left(\frac{\pi}{5}t\right) \rightarrow \frac{\pi}{5}T = 2\pi m \rightarrow T = 10m \rightarrow T = 10,\ m = 1$
Signal is periodic with $T = 10, \omega_0 = \frac{\pi}{5} \rightarrow$ FS. Signal is a sine, thus we can use the inspection method:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{e^{j\left(\frac{\pi}{5}t\right)} - e^{-j\left(\frac{\pi}{5}t\right)}}{2j} = \frac{1}{2j}e^{j\left(\frac{\pi}{5}t\right)} - \frac{1}{2j}e^{-j\left(\frac{\pi}{5}t\right)} = X[1]e^{j\omega_0 t} + X[-1]e^{-j\omega_0 t}$$

Thus, we have the FS coefficients $X[k]$ for all k: $X[k] = \begin{cases} -\frac{1}{2j}, & k = -1 \\ +\frac{1}{2j}, & k = 1 \\ 0, & otherwise \end{cases}$

f) $x(t) = Cos\left(\frac{\pi}{3}t + \frac{\pi}{5}\right) \rightarrow \frac{\pi}{3}T = 2\pi m \rightarrow T = 6m \rightarrow T = 6,\ m = 1$
Signal is periodic with $T = 6, \omega_0 = \frac{\pi}{3} \rightarrow$ FS
Since the signal is a cosine, we can use the inspection method:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{e^{j\left(\frac{\pi}{3}t+\frac{\pi}{5}\right)} + e^{-j\left(\frac{\pi}{3}t+\frac{\pi}{5}\right)}}{2} = \frac{e^{j\frac{\pi}{5}}}{2}e^{j\left(\frac{\pi}{3}t\right)} + \frac{e^{-j\frac{\pi}{5}}}{2}e^{-j\left(\frac{\pi}{3}t\right)} = X[1]e^{j\omega_0 t} + X[-1]e^{-j\omega_0 t}$$

Thus, we have the FS coefficients $X[k]$ for all k: $X[k] = \begin{cases} \frac{1}{2}e^{-j\frac{\pi}{5}}, & k = -1 \\ \frac{1}{2}e^{j\frac{\pi}{5}}, & k = 1 \\ 0, & otherwise \end{cases}$

g) $x[n]=\delta[n+2]+\delta[n-4]$. Non-periodic signal → DTFT

$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=e^{-j\Omega(-2)}+e^{-j\Omega(4)}=e^{2j\Omega}+e^{-4j\Omega}$$

h) $x[n]=\left(\frac{3}{8}\right)^n u[n-3]$. Non-periodic signal → DTFT

$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=3}^{\infty}\left(\frac{3}{8}\right)^n e^{-j\Omega n}=\sum_{n=3}^{\infty}\left(\frac{3}{8}e^{-j\Omega}\right)^n=\sum_{n=0}^{\infty}\left(\frac{3}{8}e^{-j\Omega}\right)^n-\sum_{n=0}^{2}\left(\frac{3}{8}e^{-j\Omega}\right)^n$$

$$X(e^{j\Omega})=\frac{1}{1-\frac{3}{8}e^{-j\Omega}}-\frac{1-\left(\frac{3}{8}\right)^3e^{-3j\Omega}}{1-\frac{3}{8}e^{-j\Omega}}=\frac{\left(\frac{3}{8}\right)^3e^{-3j\Omega}}{1-\frac{3}{8}e^{-j\Omega}}$$

PROBLEM 3

Use the defining equation for the DTFT to evaluate the frequency-domain representations of the following signals. You must show the procedure.

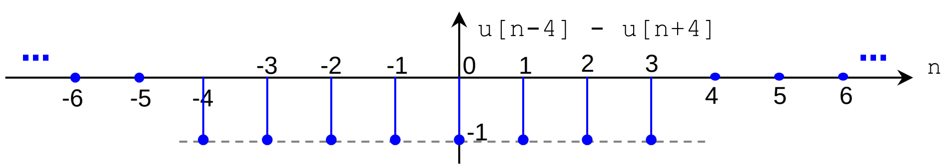
a) $x[n] = (3/5)^n (u(n-4) - u(n+4))$
b) $x[n] = b^{|n|}$, $|b| < 1$
c) $x[n] = 2\delta[5 - 3n]$
d) $x[n] = (1/4)(\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3])$
e) $x[n] = 2 + e^{-3n}$

---

a)

$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=-\infty}^{\infty}-\left(\frac{3}{5}\right)^n e^{-j\Omega n}=-\sum_{n=-4}^{3}\left(\frac{3}{5}e^{-j\Omega}\right)^n=-\frac{\left(\frac{3}{5}\right)^{-4}e^{-j\Omega(-4)}-\left(\frac{3}{5}\right)^{4}e^{-j\Omega 4}}{1-\frac{3}{5}e^{-j\Omega}}$$

$$X(e^{j\Omega})=\frac{\left(\frac{3}{5}\right)^{4}e^{-4j\Omega}-\left(\frac{3}{5}\right)^{-4}e^{4j\Omega}}{1-\frac{3}{5}e^{-j\Omega}}$$

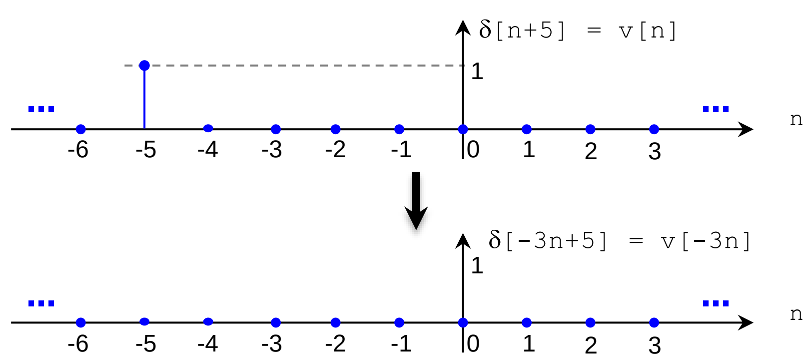


b) $x[n]=b^{|n|}, |b|<1$

$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}b^{|n|}e^{-j\Omega n}=\sum_{n=0}^{\infty}b^{n}e^{-j\Omega n}+\sum_{n=-\infty}^{-1}b^{-n}e^{-j\Omega n}=\sum_{n=0}^{\infty}\left(be^{-j\Omega}\right)^n+\sum_{n=-\infty}^{-1}\left(be^{j\Omega}\right)^{-n}$$

$$X(e^{j\Omega})=\sum_{n=0}^{\infty}\left(be^{-j\Omega}\right)^n+\sum_{k=1}^{\infty}\left(be^{j\Omega}\right)^k=\frac{1}{1-be^{-j\Omega}}+\frac{1}{1-be^{j\Omega}}-1=\frac{1-b^2}{1-2bCos\Omega+b^2}$$

c) $x[n]=2\delta[5-3n]$



$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=0$$

d) $x[n] = \frac{1}{4}(\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3])$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \frac{1}{4}e^{0} + \frac{3}{4}e^{-j\Omega} + \frac{1}{2}e^{-2j\Omega} + \frac{1}{4}e^{-3j\Omega} = \frac{1}{4} + \frac{3}{4}e^{-j\Omega} + \frac{1}{2}e^{-2j\Omega} + \frac{1}{4}e^{-3j\Omega}$$

e) $x[n] = 2 + e^{-3n}$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} 2e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} e^{-3n}e^{-j\Omega n}$$

$\sum_{n=-\infty}^{\infty} 2e^{-j\Omega n}$ tends to infinity. Thus, $X(e^{j\Omega})$ is undefined

PROBLEM 4

Determine the time-domain signals corresponding to the following DTFTs. You must show the procedure.

a) $X(e^{j\Omega}) = \sin(2\Omega) + j\cos(2\Omega)$
b) $X(e^{j\Omega}) = 3\sin(4\Omega)$
c) $X(e^{j\Omega}) = (1/2)e^{-j\Omega/2}$
d) $X(e^{j\Omega}) = \cos(\Omega) + \sin(\Omega/2)$

---

a) $X(e^{j\Omega}) = Sin(2\Omega) + jCos(2\Omega) = j\big(Cos(2\Omega) - jSin(2\Omega)\big) = je^{-j2\Omega}$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} je^{-j2\Omega}e^{j\Omega n}d\Omega = \frac{j}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n-2)}d\Omega$$

$$n = 2:\ \ x[n] = \frac{j}{2\pi}\int_{-\pi}^{\pi} 1d\Omega = j$$

$$n \neq 2:\ \ x[n] = \frac{j}{2\pi}\frac{1}{j(n-2)}e^{j\Omega(n-2)}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi(n-2)}\left(e^{j\pi(n-2)} - e^{j\pi\Omega(n-2)}\right) = \frac{1}{2\pi(n-2)}2jSin\big(\pi(n-2)\big)$$

$$Sin\big(\pi(n-2)\big) = 0, \forall n$$

Thus: $x[n] = j\delta[n-2]$

b) $X(e^{j\Omega}) = 3Sin(4\Omega) = \frac{3}{2j}e^{j4\Omega} - \frac{3}{2j}e^{-j4\Omega}$

We use the time-shift property of the DTFT along with the fact that the DTFT of an impulse is 1.

$$x[n-k] \overset{DTFT}{\longleftrightarrow} e^{-jk\Omega}X(e^{j\Omega}), \qquad \delta[n] \overset{DTFT}{\longleftrightarrow} 1$$

And we determine that:

$$\delta[n-k] \overset{DTFT}{\longleftrightarrow} e^{-jk\Omega}$$

In exercise 4(a), we demonstrate that the DTFT of $j\delta[n-2]$ is $je^{-j2\Omega}$

Finally: $x[n] = \frac{3}{2j}\delta[n+4] - \frac{3}{2j}\delta[n-4]$

c) $X(e^{j\Omega}) = \frac{1}{2}e^{-j\frac{\Omega}{2}}, -\pi \leq \Omega < \pi$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}e^{-j\frac{\Omega}{2}}e^{j\Omega n}d\Omega = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{j\left(n-\frac{1}{2}\right)\Omega}d\Omega = \frac{1}{4\pi}\frac{1}{j\left(n-\frac{1}{2}\right)}e^{j\left(n-\frac{1}{2}\right)\Omega}\Big|_{-\pi}^{\pi}$$

$$x[n] = \frac{1}{4\pi}\frac{1}{j\left(n-\frac{1}{2}\right)}\left(e^{j\left(n-\frac{1}{2}\right)\pi} - e^{-j\left(n-\frac{1}{2}\right)\pi}\right), e^{j\frac{\pi}{2}} = j, e^{-j\frac{\pi}{2}} = -j$$

$$x[n] = -\frac{1}{4\pi}\frac{1}{j\left(n-\frac{1}{2}\right)}(je^{jn\pi} + je^{-jn\pi}) = -\frac{1}{4\pi\left(n-\frac{1}{2}\right)}(e^{jn\pi} + e^{-jn\pi}) = -\frac{1}{4\pi\left(n-\frac{1}{2}\right)}2Cos(n\pi)$$

$$x[n] = -\frac{(-1)^n}{2\pi\left(n-\frac{1}{2}\right)}$$

d) $X(e^{j\Omega}) = Cos(\Omega) + Sin\left(\frac{\Omega}{2}\right), \ -\pi \le \Omega < \pi$

$$X(e^{j\Omega}) = \frac{e^{j\Omega}}{2} + \frac{e^{-j\Omega}}{2} + \frac{e^{j\frac{\Omega}{2}}}{2j} - \frac{e^{-j\frac{\Omega}{2}}}{2j}$$

$$x[n] = \frac{\delta[n+1]}{2} + \frac{\delta[n-1]}{2} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{j\frac{\Omega}{2}} e^{j\Omega n} d\Omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{-j\frac{\Omega}{2}} e^{j\Omega n} d\Omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{-j\frac{\Omega}{2}} e^{j\Omega n} d\Omega = -\frac{(-1)^n}{2\pi j\left(n-\frac{1}{2}\right)}, using\ result\ from\ exercise\ 4(c)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{j\frac{\Omega}{2}} e^{j\Omega n} d\Omega = \frac{1}{4\pi j}\int_{-\pi}^{\pi} e^{j\left(n+\frac{1}{2}\right)\Omega} d\Omega = \frac{1}{4\pi}\frac{1}{j\left(n+\frac{1}{2}\right)} e^{j\left(n+\frac{1}{2}\right)\Omega}\Big|_{-\pi}^{\pi}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{j\frac{\Omega}{2}} e^{j\Omega n} d\Omega = \frac{1}{4\pi}\frac{1}{j\left(n+\frac{1}{2}\right)}\left(e^{j\left(n+\frac{1}{2}\right)\pi} - e^{-j\left(n+\frac{1}{2}\right)\pi}\right), e^{j\frac{\pi}{2}} = j, e^{-j\frac{\pi}{2}} = -j$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{j\frac{\Omega}{2}} e^{j\Omega n} d\Omega = \frac{1}{4\pi}\frac{1}{j\left(n+\frac{1}{2}\right)}\left(je^{jn\pi} + je^{-jn\pi}\right) = \frac{1}{4\pi\left(n+\frac{1}{2}\right)}\left(e^{jn\pi} + e^{-jn\pi}\right) = \frac{1}{4\pi\left(n+\frac{1}{2}\right)} 2Cos(n\pi)$$

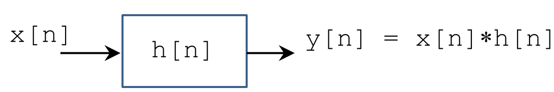
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2j} e^{j\frac{\Omega}{2}} e^{j\Omega n} d\Omega = \frac{(-1)^n}{2\pi\left(n+\frac{1}{2}\right)}$$

Finally:

$$x[n] = \frac{\delta[n+1]}{2} + \frac{\delta[n-1]}{2} + \frac{(-1)^n}{2\pi j\left(n+\frac{1}{2}\right)} + \frac{(-1)^n}{2\pi j\left(n-\frac{1}{2}\right)}$$

PROBLEM 5

The following LTI system has an input described by:



`x[n] = sin((5π/7)n + π/8)`

The Fourier representation of the impulse response `h[n]` is given by: `H[k] = ke`$^{-k}$, on `0 ≤ k ≤ N-1`.

a) Determine the period 'N' of the signal `x[n]`.
b) Determine the DTFS coefficients `X[k]`.
c) Obtain the frequency domain representation `Y[k]` of the output signal `y[n]`.
d) Using MATLAB, plot `X[k]`, `H[k]`, and `Y[k]` for three periods. Plot the magnitude and the phase spectra.

---

a) $x[n] = Sin\left(\frac{5\pi}{7}n + \frac{\pi}{8}\right) \ \rightarrow \frac{5\pi}{7}N = 2\pi m \ \rightarrow N = \frac{14}{5}m \ \rightarrow N = 14, \ m = 5 \ \rightarrow \Omega_0 = \frac{\pi}{7}$

b) Since the signal is a sine, we can use the inspection method:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} = \frac{e^{j\left(\frac{5\pi}{7}n+\frac{\pi}{8}\right)} - e^{-j\left(\frac{5\pi}{7}n+\frac{\pi}{8}\right)}}{2j} = \frac{e^{j\frac{\pi}{8}}}{2j} e^{j\left(\frac{5\pi}{7}n\right)} - \frac{e^{j\frac{\pi}{8}}}{2j} e^{-j\left(\frac{5\pi}{7}n\right)}$$

$$x[n] = \frac{e^{j\frac{\pi}{8}}}{2j} e^{j(5\Omega_0 n)} - \frac{e^{-j\frac{\pi}{8}}}{2j} e^{-j(5\Omega_0 n)} = X[5] e^{j(5\Omega_0 n)} + X[-5] e^{-j(5\Omega_0 n)}$$

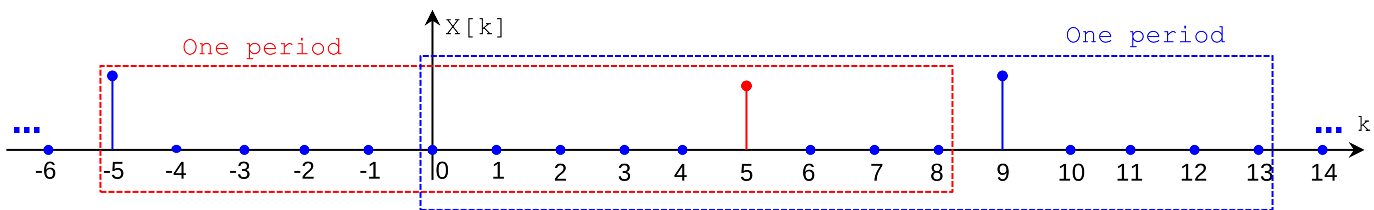
Thus, we have the DTFS coefficients $X[k]$ for one period ($k = -5\ to\ 8$):

$$X[k] = \begin{cases} -\dfrac{e^{-j\frac{\pi}{8}}}{2j}, & k = -5 \\ \dfrac{e^{j\frac{\pi}{8}}}{2j}, & k = 5 \\ 0, & k = -4,-3,-2,-1,0,1,2,3,4,6,7,8 \end{cases}$$

c) $H[k] = ke^{-k}$, on $0 \le k \le 13$

$H[k]$ is defined for a different range. We can either redefine $H[k]$ from -4 to 8, or $X[k]$ from 0 to 13. Let's redefine $X[k]$ from 0 to 13:

$$X[k] = \begin{cases} \dfrac{e^{j\frac{\pi}{8}}}{2j}, & k = 5 \\ -\dfrac{e^{-j\frac{\pi}{8}}}{2j}, & k = 9 \\ 0, & k = 0,1,2,3,4,6,7,8,10,11,12,13 \end{cases}$$



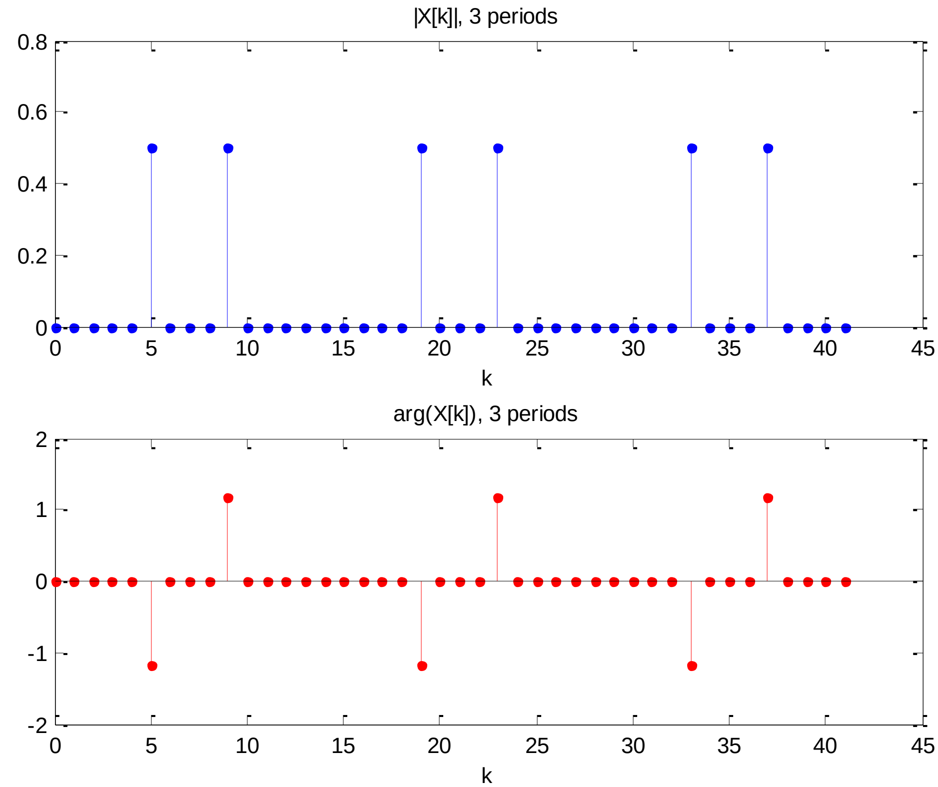
Now, using the convolution property, we get the DTFS for one period: $0 \le k \le 13$

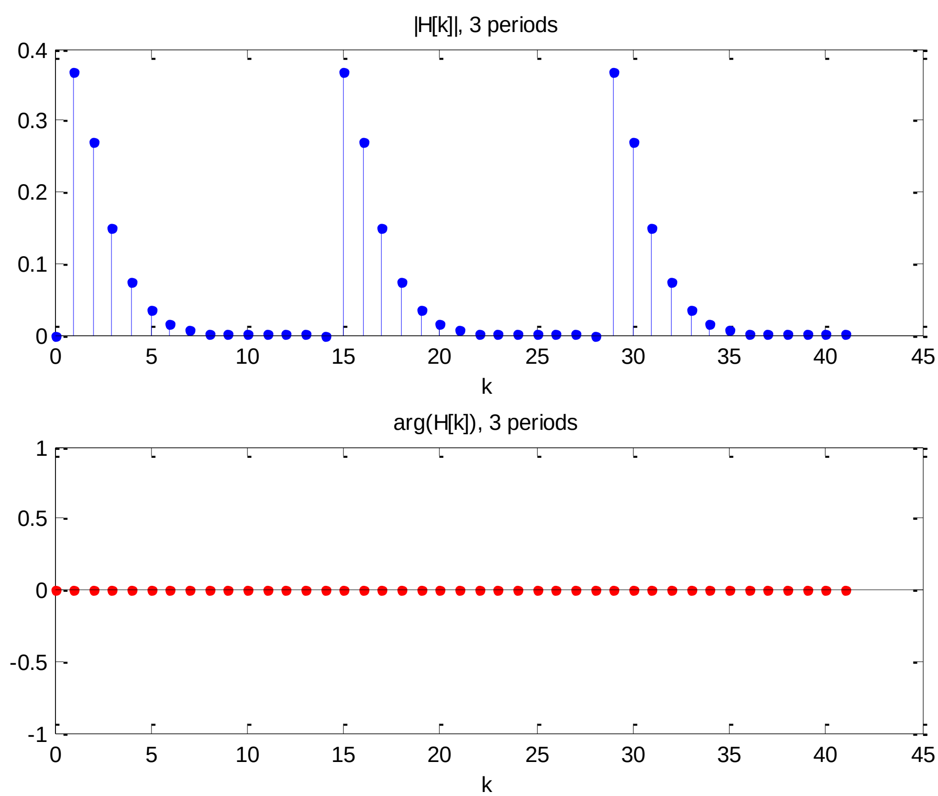
$$Y[k] = NX[k]H[k] = \begin{cases} 14\dfrac{e^{j\frac{\pi}{8}}}{2j}5e^{-5}, & k = 5 \\ -14\dfrac{e^{-j\frac{\pi}{8}}}{2j}9e^{-9}, & k = 9 \\ 0, & k = 0,1,2,3,4,6,7,8,10,11,12,13 \end{cases}$$

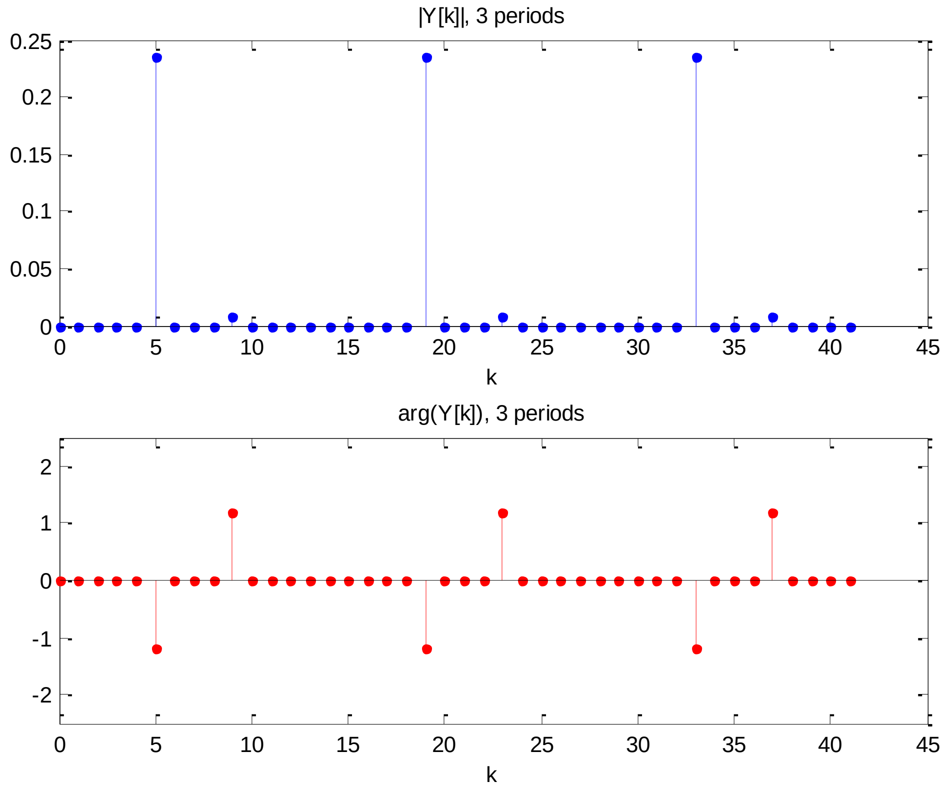
d) MATLAB:

```
clear all; close all; clc

k = 0:13; k_3p = 0:41;

% In the period 0 to 13:
H = k.*exp(-k);
X(1:14) = 0;
X(6) = (1/2i)*exp(1i*pi/8);
X(10) = -(1/2i)*exp(-1i*pi/8);

Y(1:14) = 0;
Y(6) = (1/2i)*exp(1i*pi/8)*14*5*exp(-5);
Y(10) = -(1/2i)*exp(-1i*pi/8)*14*9*exp(-9);

X_3p = [X X X]; % Here, we generate replicas
H_3p = [H H H]; % Here, we generate replicas
Y_3p = [Y Y Y]; % Here, we generate replicas

figure;
subplot (2,1,1), stem(k_3p, abs(X_3p),'.b'); axis ([0 45 0 0.8]);
set(gca, 'Fontsize',8); xlabel('k'); title ('|X[k]|, 3 periods');
subplot (2,1,2), stem(k_3p, angle(X_3p),'.r');
set(gca, 'Fontsize',8); xlabel('k'); title ('arg(X[k]), 3 periods');

figure;
subplot (2,1,1), stem(k_3p, abs(H_3p),'.b');
set(gca, 'Fontsize',8); xlabel('k'); title ('|H[k]|, 3 periods');
subplot (2,1,2), stem(k_3p, angle(H_3p),'.r');
set(gca, 'Fontsize',8); xlabel('k'); title ('arg(H[k]), 3 periods');
```

```
figure;
subplot (2,1,1), stem(k_3p, abs(Y_3p),'.b');
set(gca, 'Fontsize',8); xlabel('k'); title ('|Y[k]|, 3 periods');
subplot (2,1,2), stem(k_3p, angle(Y_3p),'.r'); axis ([0 45 -2.5 2.5]);
set(gca, 'Fontsize',8); xlabel('k'); title ('arg(Y[k]), 3 periods');
```







PROBLEM 6

Use the properties of Fourier representation (e.g., time-differentiation, convolution, time-shift, frequency-shift) to find the FT of:

$$y(t) = \frac{d}{dt}\{e^{-4t}u(t-5) * e^{-2t}u(t-3)\}$$

Note: '$*$' denotes convolution.

Hint: It might help you that the FT of $e^{-at}u(t)$ is $1/(a+j\omega)$

$w(t) = e^{-4t}u(t-5), v(t) = e^{-2t}u(t-3),\quad z(t) = w(t) * v(t),\quad y(t) = \frac{d}{dt}z(t)$

<u>Differentiation Property of FT</u>: If $z(t) \overset{FT}{\longleftrightarrow} Z(j\omega)$, then: $\frac{d}{dt}z(t) \overset{FT}{\longleftrightarrow} j\omega Z(j\omega)$

Then: $Y(j\omega) = j\omega Z(j\omega)$

<u>Convolution Property of FT</u>: $z(t) = w(t) * v(t) \overset{FT}{\longleftrightarrow} Z(j\omega) = W(j\omega)V(j\omega)$

Then: $Y(j\omega) = j\omega W(j\omega)V(j\omega)$

<u>Knowledge of a common FT pair</u>: $s(t) = e^{-at}u(t) \overset{FT}{\longleftrightarrow} S(j\omega) = \frac{1}{a+j\omega}$

Then: $w(t) = e^{-4t}u(t-5) = e^{-20}e^{-4(t-5)}u(t-5) = e^{-20}r(t-5)$

where: $r(t) = e^{-4t}u(t)$, and $R(j\omega) = \frac{1}{4+j\omega}$

Also: $v(t) = e^{-2t}u(t-3) = e^{-6}e^{-2(t-3)}u(t-3) = e^{-6}p(t-3)$

where: $p(t) = e^{-2t}u(t)$, and $P(j\omega) = \frac{1}{2+j\omega}$

<u>Time Shift Property of FT</u>: If $r(t) \overset{FT}{\longleftrightarrow} R(j\omega)$, then: $r(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0}R(j\omega)$

Then: $W(j\omega) = e^{-20}e^{-j5\omega}R(j\omega) = \frac{e^{-20}e^{-j5\omega}}{4+j\omega}$, $V(j\omega) = e^{-6}e^{-j3\omega}P(j\omega) = \frac{e^{-6}e^{-j3\omega}}{2+j\omega}$

Finally:

$$X(j\omega) = j\omega W(j\omega)V(j\omega) = \frac{e^{-26}j\omega e^{-8j\omega}}{(2+j\omega)(4+j\omega)}$$

PROBLEM 7

Given the following DTFS pair ($\Omega = \pi/10$):

$$x[n] = \frac{sin\left(\frac{11\pi}{20}n\right)}{sin\left(\frac{\pi}{20}n\right)} \xleftrightarrow{DTFS;\ \pi/10} X[k]$$

Evaluate the time-domain signal `y[n]` for the following DTFS coefficients `Y[k]`. These DTFS coefficients `Y[k]` happen to have a relationship with the DTFS coefficients `X[k]`. You can use properties of the DTFS.

a) `Y[k] = (1/2)(X[k-4] + X[k+4])`
b) `Y[k] = 3X[k-2]`
c) `Y[k] = X[k]` $(*)$`X[k], where` $(*)$ denotes periodic convolution.

a) We use the frequency shift property:

$$e^{jk_0\Omega n}x[n] \xleftrightarrow{DTFS,\ \Omega} X[k-k_0]$$

$$Y[k] = \frac{1}{2}(X[k-4] + X[k+4])$$

$$\rightarrow y[n] = \frac{1}{2}\left(e^{j4\frac{\pi}{10}n}x[n] + e^{-j4\frac{\pi}{10}n}x[n]\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}\right)x[n] = Cos\left(\frac{2\pi}{5}n\right)x[n]$$

$$y[n] = Cos\left(\frac{2\pi}{5}n\right)\frac{sin\left(\frac{11\pi}{20}n\right)}{sin\left(\frac{\pi}{20}n\right)}$$

b) Here, we also use the frequency shift property:

$$Y[k] = 3X[k-2]$$

$$\rightarrow y[n] = 3e^{j2\frac{\pi}{10}n}x[n] = 3e^{j\frac{\pi}{5}n}x[n]$$

$$y[n] = 3e^{j\frac{\pi}{5}n}\frac{sin\left(\frac{11\pi}{20}n\right)}{sin\left(\frac{\pi}{20}n\right)}$$

c) Here, se use the multiplication property:

$$v[n]w[n] \xleftrightarrow{DTFS,\ \Omega} V[k](*)W[k]$$

$$\rightarrow y[n] = x[n]x[n] = (x[n])^2$$

$$y[n] = \frac{sin^2\left(\frac{11\pi}{20}n\right)}{sin^2\left(\frac{\pi}{20}n\right)}$$