Solutions - Homework # 2

PROBLEM 1

Evaluate the DT convolution: `y[n] = x[n]*h[n]` for the following cases:

a)	`x[n] = u[n] - u[n-8]`	`h[n] = (1/4)(u[n] - u[n-5])`
b)	`x[n] = 3^n u[-n+4]`	`h[n] = u[n-3]`
c)	`x[n] = u[n+2]`	`h[n] = u[n-2]`
d)	`x[n] = sin(πn)u[n]`	`h[n] = u[n-1]`
e)	`x[n] = (1/2)^n u[n]`	`h[n] = u[n+1]`
f)	`x[n] = u[n+10] - 2u[n]`	`h[n] = α^n u[n], |α| < 1`

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a) `x[n] = u[n] - u[n-8]` `h[n] = (1/4)(u[n] - u[n-5])`

- $0 \le n \le 4$: $w_n[k] = \frac{1}{4},\ 0 \le k \le n,\ \rightarrow\ y[n] = \sum_0^n 1/4 = \frac{1}{4}(n+1),\ 0 \le n \le 4$
- $5 \le n \le 7$: $w_n[k] = \frac{1}{4},\ n-4 \le k \le n,\ \rightarrow\ y[n] = \sum_{n-4}^n 1/4 = \frac{5}{4},\ 5 \le n \le 7$
- $8 \le n \le 11$: $w_n[k] = \frac{1}{4},\ n-4 \le k \le 7,\ \rightarrow\ y[n] = \sum_{n-4}^7 1/4 = \frac{1}{4}(12-n), 8 \le n \le 11$
- $n < 0\ or\ n > 11$: $w_n[k] = 0, \forall k,\ \rightarrow\ y[n] = 0, n < 0\ or\ n > 11$

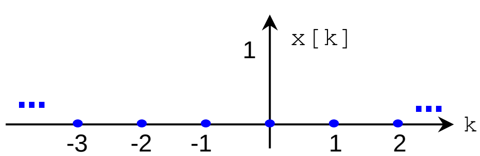
b) `x[n] = 3^n u[-n+4]` `h[n] = u[n-3]`

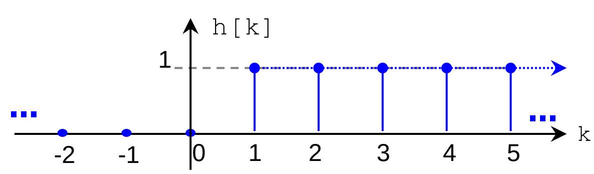
- $-\infty < n \le 7$: $w_n[k] = 3^k, -\infty \le k \le n-3,\ \rightarrow\ y[n] = \sum_{-\infty}^{n-3} 3^k = \left(\frac{3}{2}\right) 3^{n-3},\ -\infty < n \le 7$
- $8 \le n < \infty$: $w_n[k] = 3^k, -\infty \le k \le 4,\ \rightarrow\ y[n] = \sum_{-\infty}^{4} 3^k = \left(\frac{3}{2}\right) 3^4,\ 8 \le n < \infty$

c) `x[n] = u[n+2]` `h[n] = u[n-2]`

- $0 \le n < \infty$: $w_n[k] = 1, -2 \le k \le n-2,\ \rightarrow\ y[n] = \sum_{-2}^{n-2} 1 = n+1,\ 0 \le n < \infty$:
- $n < 0$: $w_n[k] = 0, \forall k,\ \rightarrow\ y[n] = 0, n < 0$

d) `x[n] = sin(πn)u[n]` `h[n] = u[n-1]`





- $y[n] = 0,\ \forall n$

e) `x[n] = (1/2)^n u[n]` `h[n] = u[n+1]`

- $-1 \le n \le \infty$: $w_n[k] = (0.5)^k,\ 0 \le k \le n+1,\ \rightarrow\ y[n] = \sum_0^{n+1} 0.5^k = \frac{1-0.5^{n+2}}{0.5}, -1 \le n \le \infty$
- $n < -1$: $w_n[k] = 0, \forall k,\ \rightarrow\ y[n] = 0,\ n < -1$

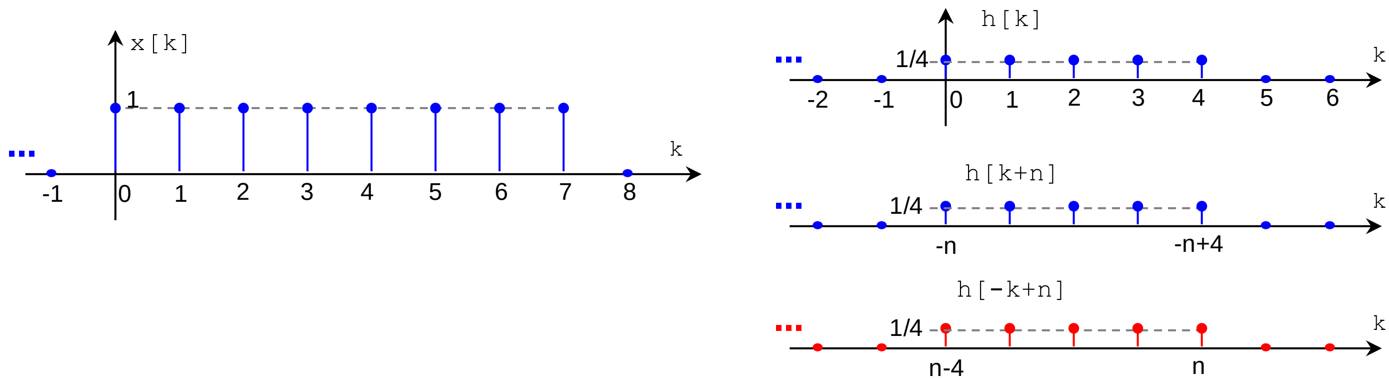
f) `x[n] = u[n+10] - 2u[n]` `h[n] = α^n u[n], |α| < 1`

- $-10 \le n \le -1$: $w_n[k] = \alpha^{n-k},\ -10 \le k \le n, \rightarrow y[n] = \sum_{-10}^{n} \alpha^{n-k} = \alpha^n \frac{\alpha^{10} - \alpha^{-(n+1)}}{1-\alpha^{-1}}, -10 \le n \le -1$
- $n \ge 0$: $w_n[k] = \alpha^{n-k},\ -10 \le k \le -1.\ w_n[k] = -\alpha^{n-k},\ 0 \le k \le n$

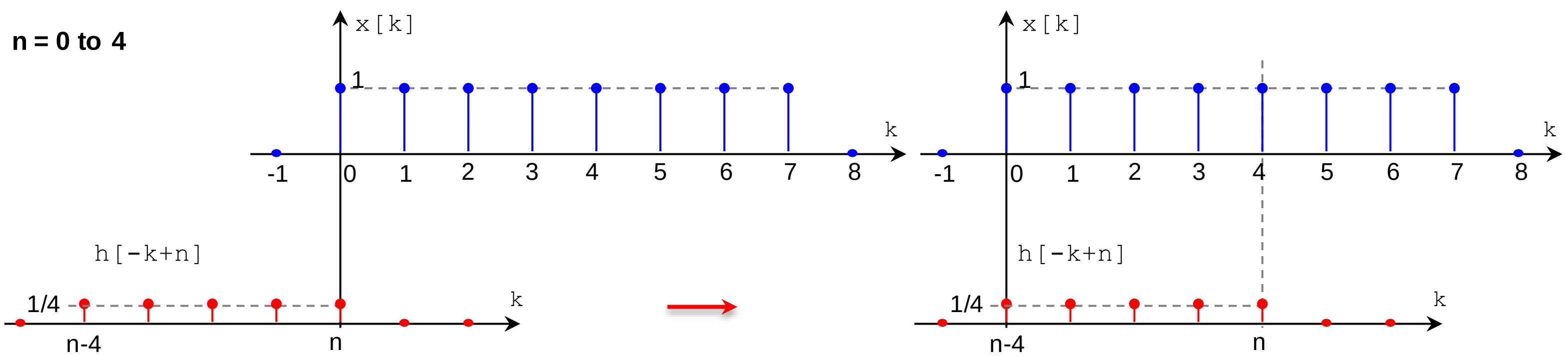
$$\rightarrow y[n] = \sum_{-10}^{-1} \alpha^{n-k} - \sum_{0}^{n} \alpha^{n-k} = \alpha^n \frac{\alpha^{10} - \alpha^0}{1-\alpha^{-1}} - \alpha^n \frac{\alpha^0 - \alpha^{-(n+1)}}{1-\alpha^{-1}} = \alpha^n \frac{\alpha^{10} - 2 - \alpha^{-(n+1)}}{1-\alpha^{-1}}, n \ge 0$$

- $n < -10$: $w_n[k] = 0, \forall k,\ \rightarrow\ y[n] = 0,\ n < -10$

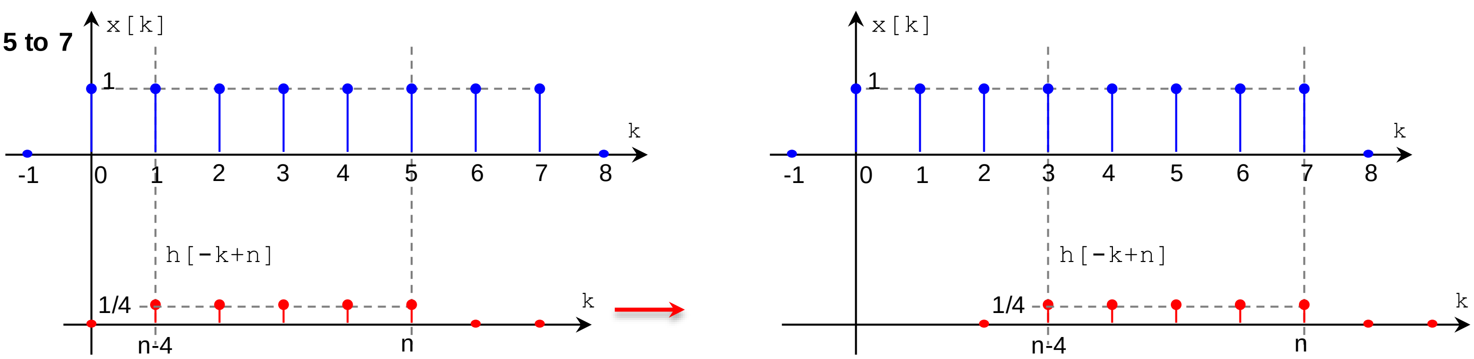
a)



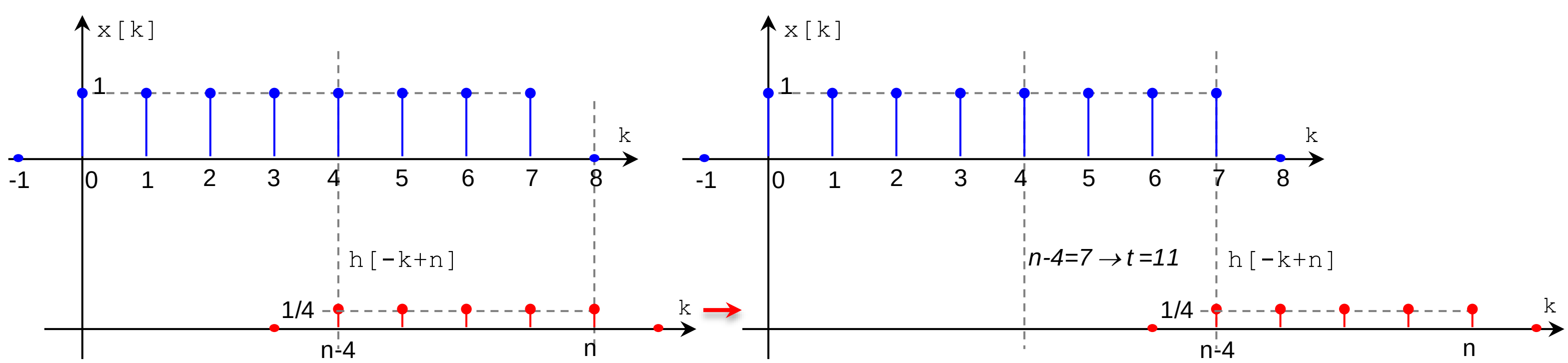
n = 0 to 4



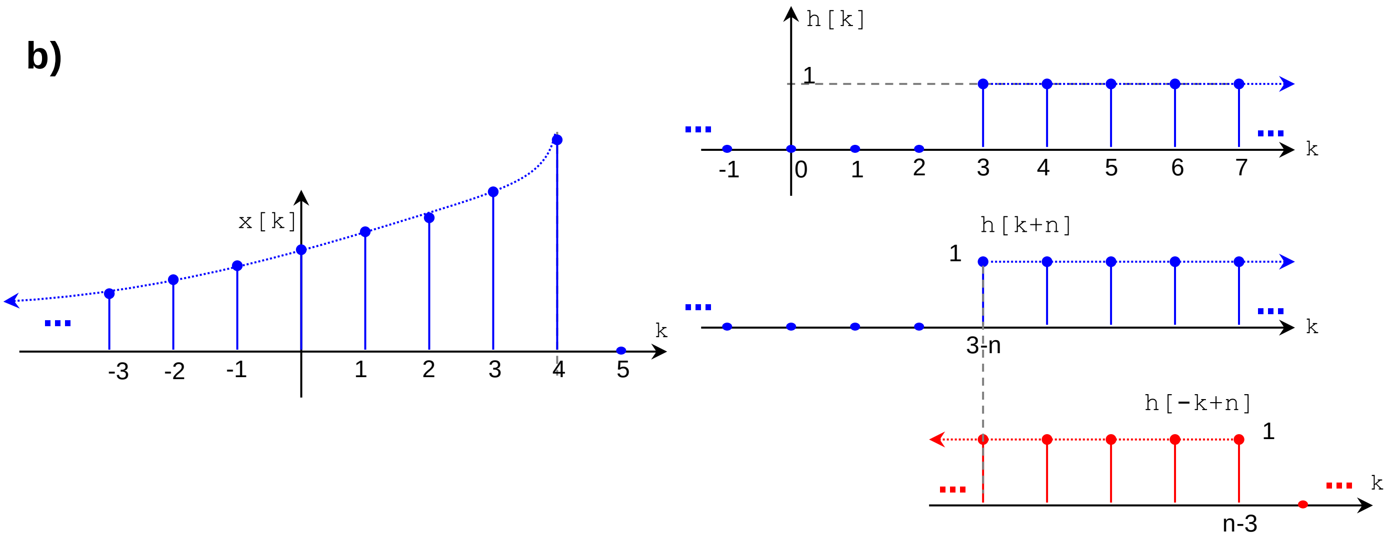
n = 5 to 7



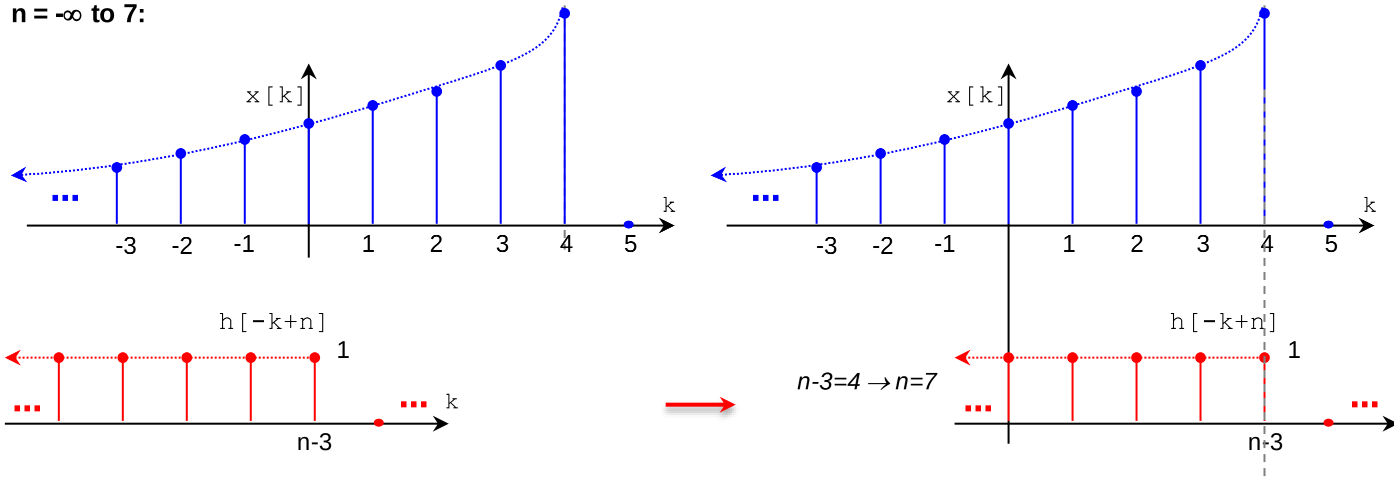
n = 8 to 11



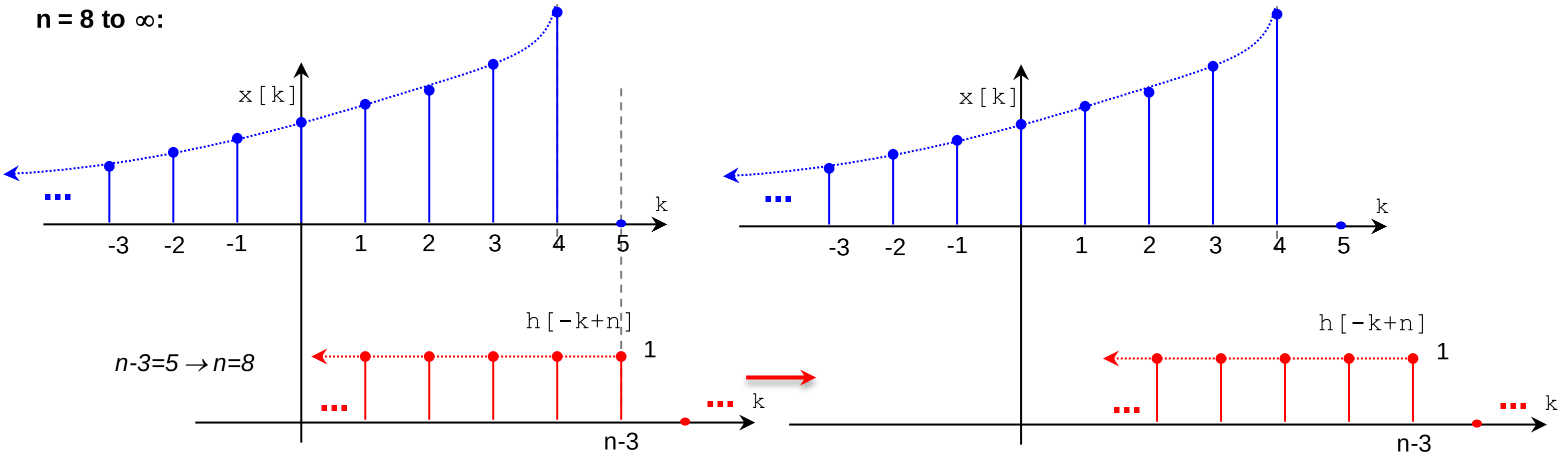
b)


h[k]
1
-1
0
1
2
3
4
5
6
7
k
x[k]
-3
-2
-1
1
2
3
4
5
k
h[k+n]
1
3-n
k
h[-k+n]
1
k
n-3

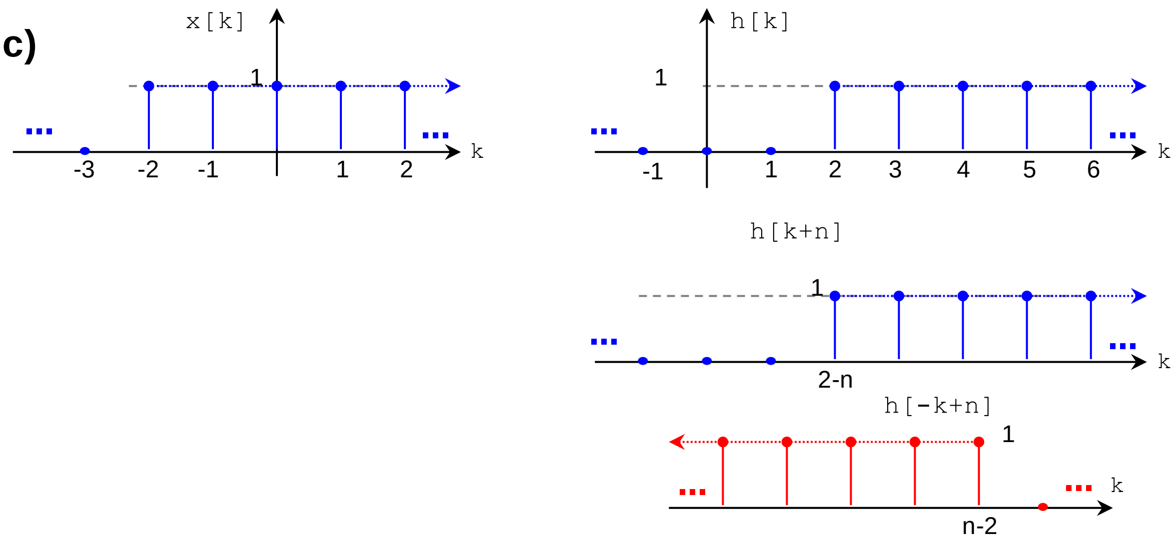
n = -∞ to 7:


x[k]
-3
-2
-1
1
2
3
4
5
k
h[-k+n]
1
k
n-3
n-3=4 → n=7

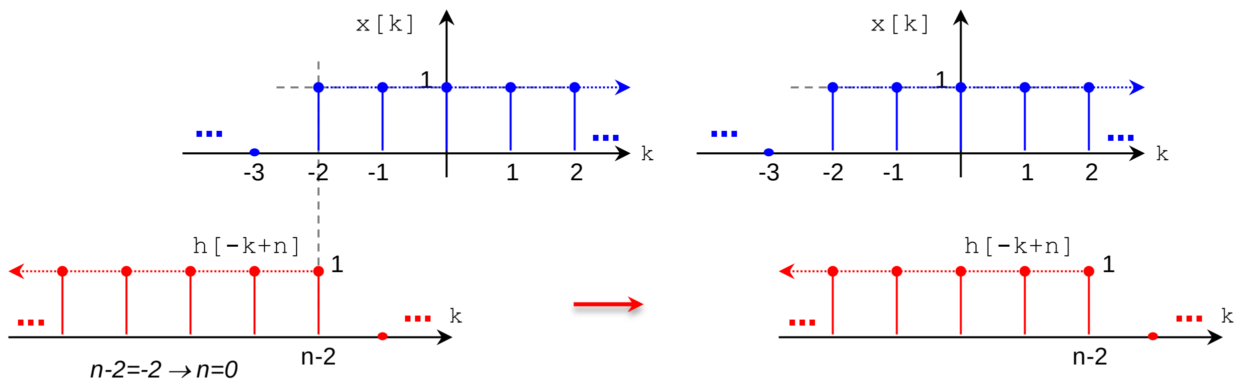
n = 8 to ∞:


x[k]
-3
-2
-1
1
2
3
4
5
k
h[-k+n]
1
k
n-3
n-3=5 → n=8

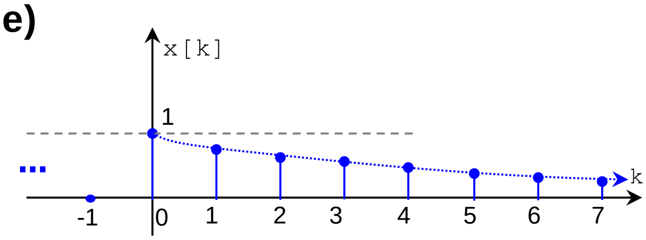
c)

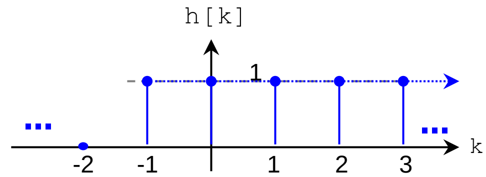


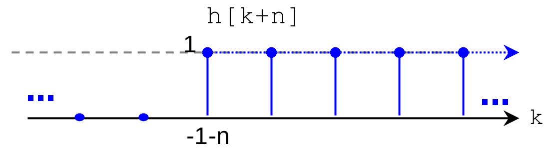
n = 0 to ∞:

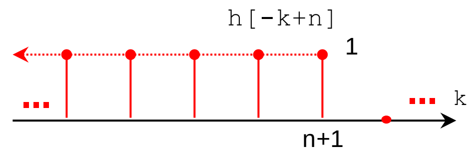


e)

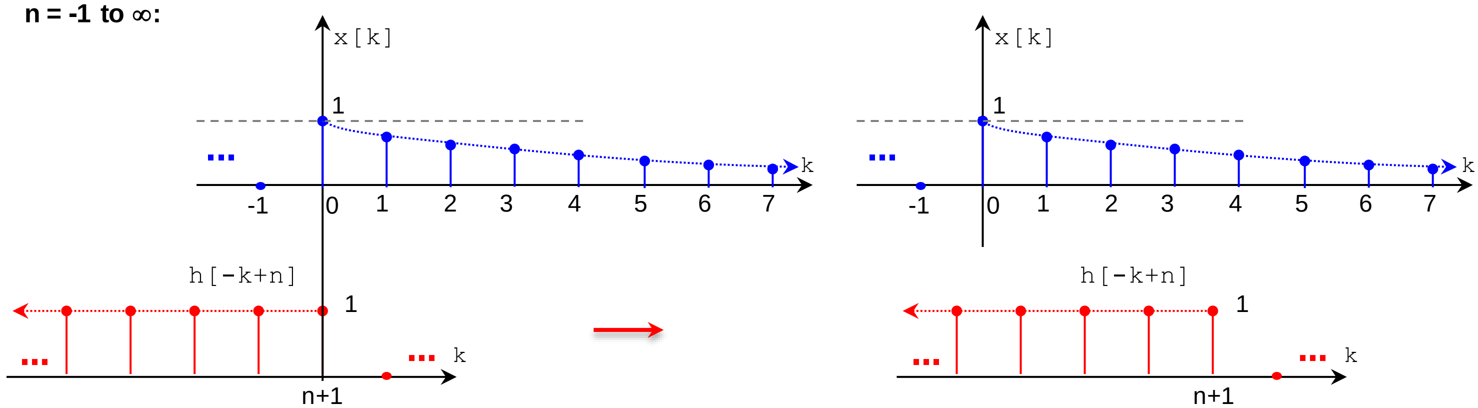

x[k]
1
...
k
-1
0
1
2
3
4
5
6
7


h[k]
1
...
...
k
-2
-1
1
2
3

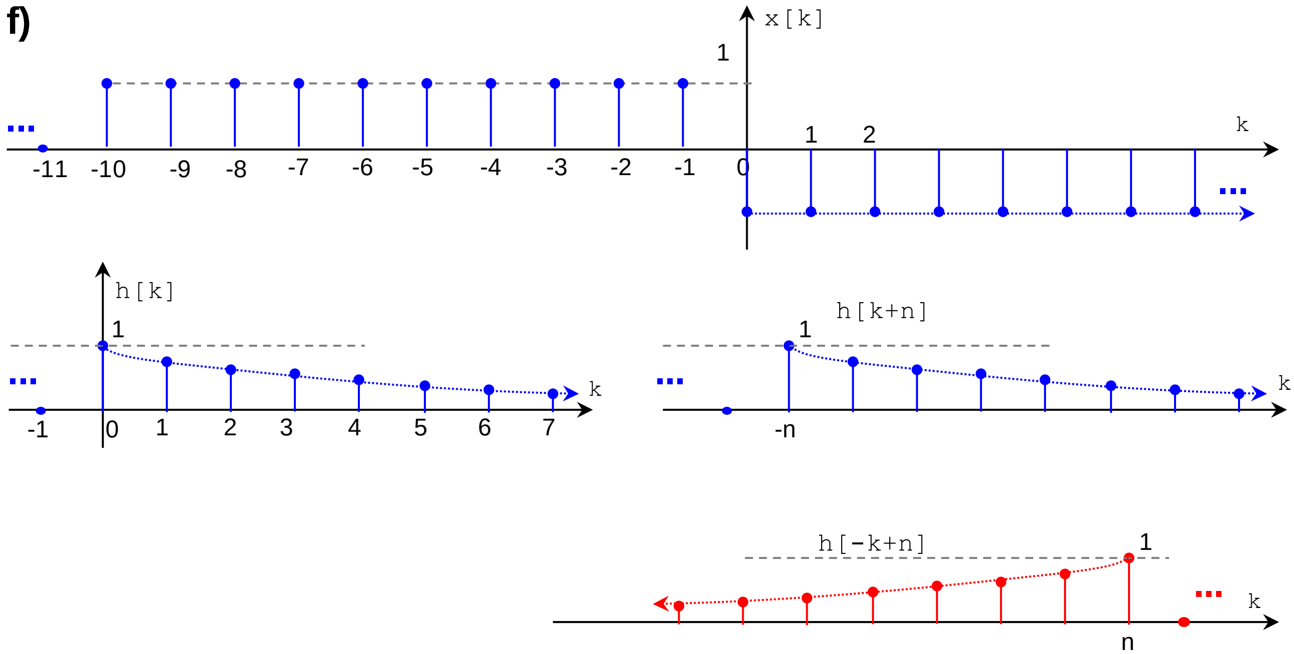

h[k+n]
1
...
...
k
-1-n


h[-k+n]
1
...
...
k
n+1

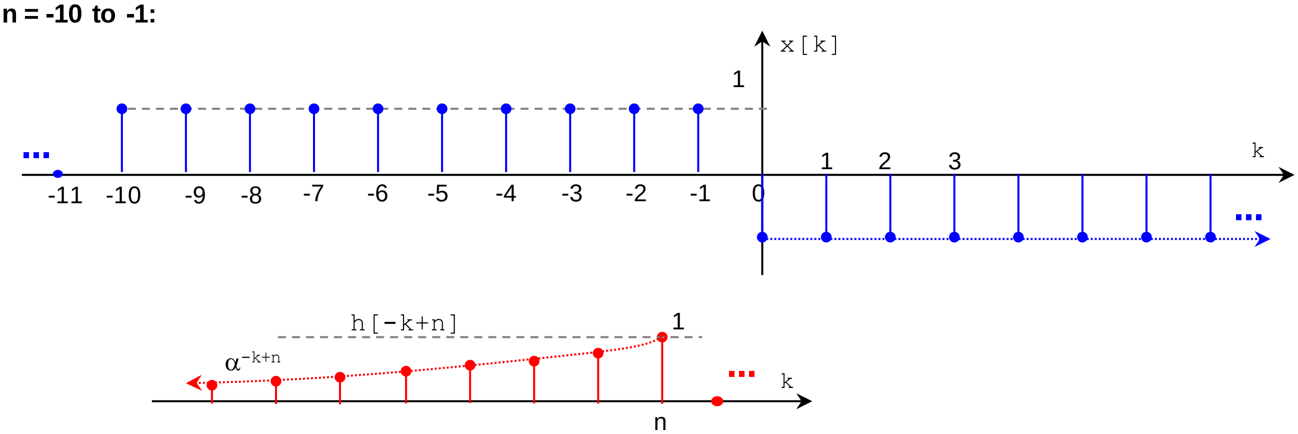
n = -1 to ∞:


x[k]
1
...
k
-1
0
1
2
3
4
5
6
7
h[-k+n]
1
...
...
k
n+1
x[k]
1
...
k
-1
0
1
2
3
4
5
6
7
h[-k+n]
1
...
...
k
n+1

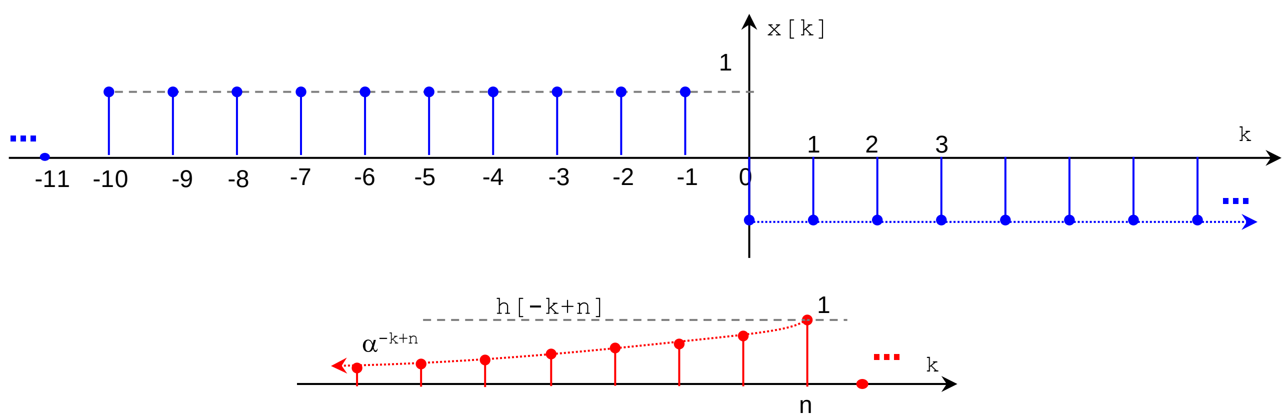
f)



n = -10 to -1:



n = 0 to ∞:

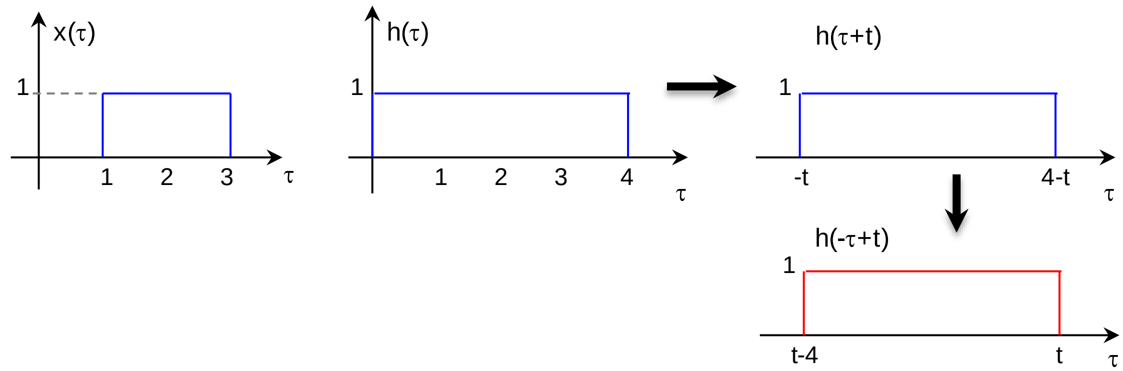


PROBLEM 2

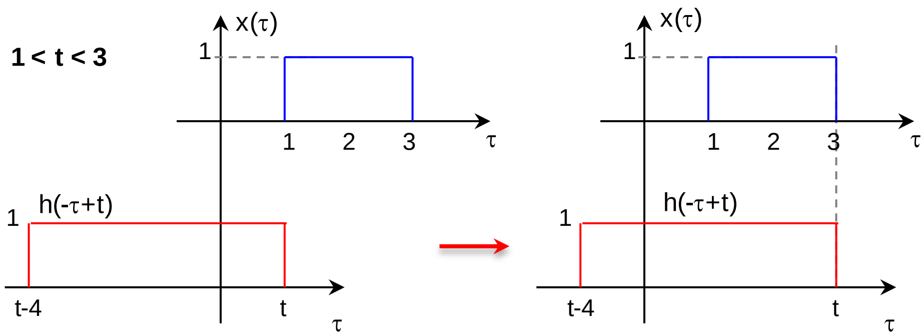
Evaluate the CT convolution: `y(t) = x(t)*h(t)` for the following cases:

```
a) x(t) = u(t-1) - u(t-3)                  h(t) = u(t) - u(t-4)
b) x(t) = sin(πt)(u(t+1) - u(t-1))         h(t) = u(t) - u(t-3)
c) x(t) = e^-3t u(t)                        h(t) = u(t+2)
d) x(t) = e^-2t (u(t+2) - u(t-2))           h(t) = u(t) - u(t-2)
```

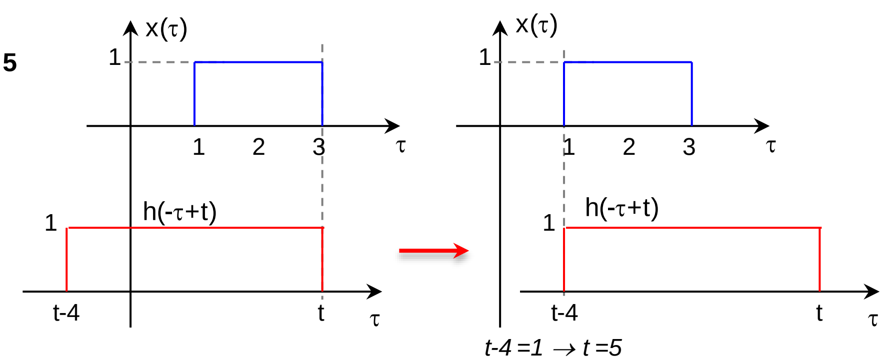
```
a) x(t) = u(t-1) - u(t-3)                  h(t) = u(t) - u(t-4)
```



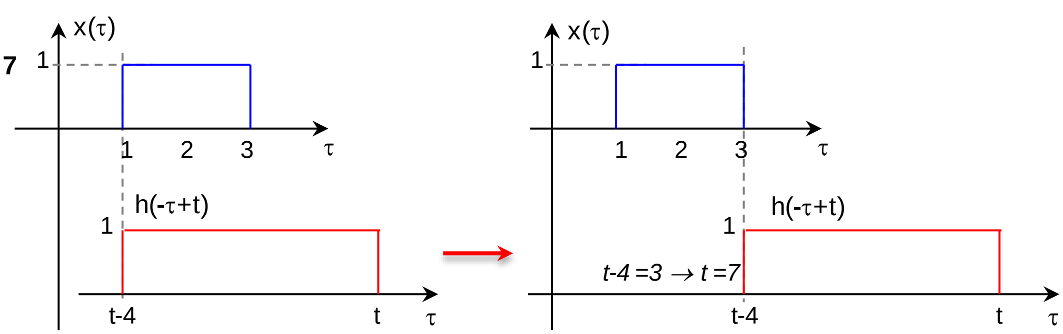
1 < t < 3



3 < t < 5

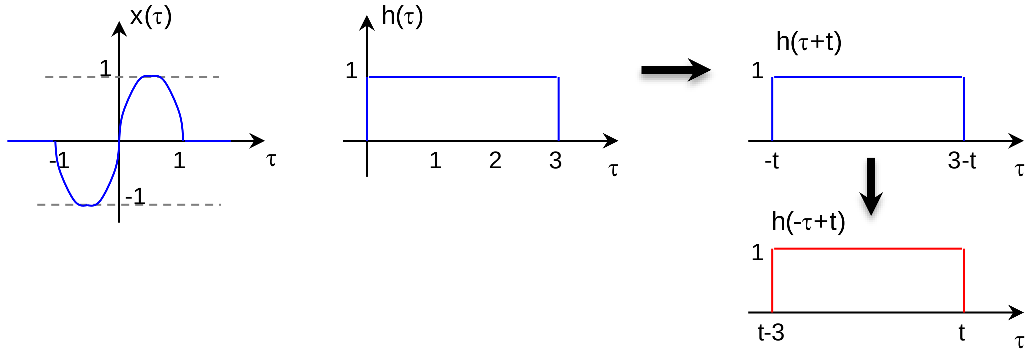


5 < t < 7

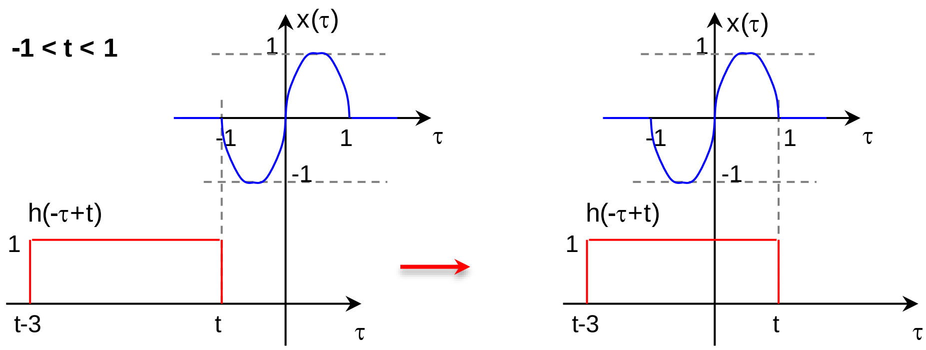


- $1 < t < 3$: $w_t(\tau) = 1,\ 1 < \tau < t,\ \rightarrow\ y(t) = \int_1^t 1d\tau = t - 1,\ 1 < t < 3$
- $3 < t < 5$: $w_t(\tau) = 1,\ 1 < \tau < 3,\ \rightarrow\ y(t) = \int_1^3 1d\tau = 2,\ 3 < t < 5$
- $5 < t < 7$: $w_t(\tau) = 1,\ t - 4 < \tau < 3,\ \rightarrow\ y(t) = \int_{t-4}^3 1d\tau = 7 - t,\ 5 < t < 7$
- $t < 1\ or\ t > 7$: $w_t(\tau) = 0,\ \forall\tau,\ \rightarrow\ y(t) = 0,\ t < 1\ or\ t > 7$

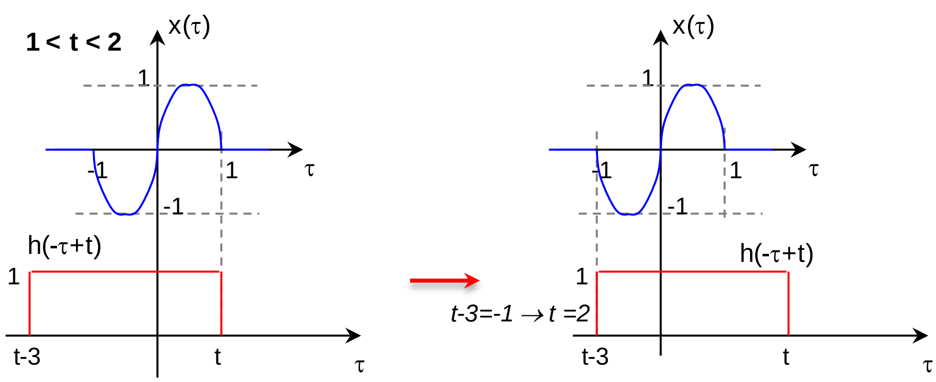
b) x(t) = sin(πt)(u(t+1) - u(t-1))   h(t) = u(t) - u(t-3)



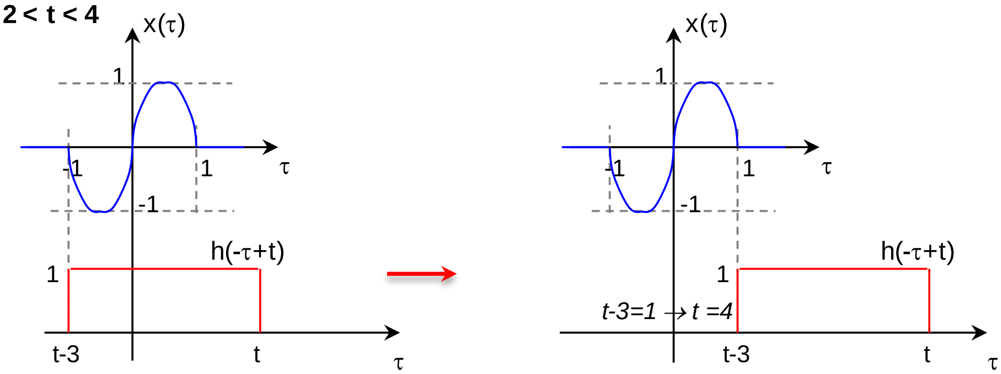
-1 < t < 1



1 < t < 2

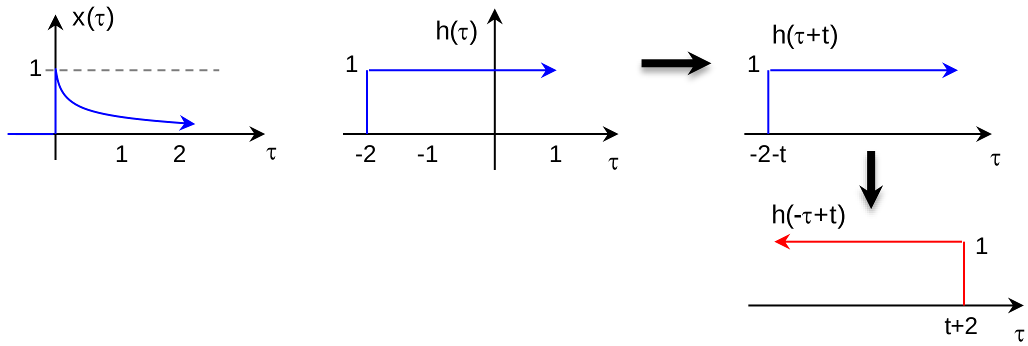


2 < t < 4

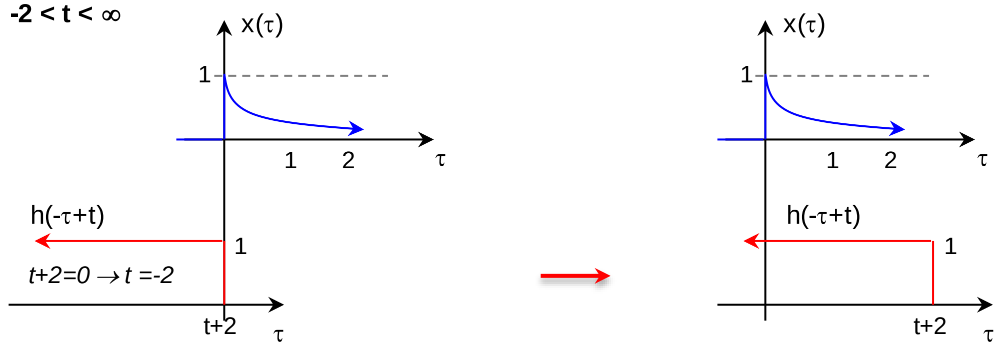


- $-1 < t < 1$: $w_t(\tau) = \sin(\pi\tau), -1 < \tau < t, \rightarrow y(t) = \int_{-1}^{t} \sin(\pi\tau)d\tau = -\left(\frac{\cos(\pi t)+1}{\pi}\right), -1 < t < 1$
- $1 < t < 2$: $w_t(\tau) = \sin(\pi\tau), -1 < \tau < 1, \rightarrow y(t) = \int_{-1}^{1} \sin(\pi\tau)d\tau = 0, 1 < t < 2$
- $2 < t < 4$: $w_t(\tau) = \sin(\pi\tau), t-3 < \tau < 1, \rightarrow y(t) = \int_{t-3}^{1} \sin(\pi\tau)d\tau = \left(\frac{\cos(\pi(t-3))+1}{\pi}\right), 2 < t < 4$
- $t < -1$ or $t > 4$: $w_t(\tau) = 0, \forall\tau, \rightarrow y(t) = 0, t < -1$ or $t > 4$

c) $x(t) = e^{-3t}u(t)$ $h(t) = u(t+2)$



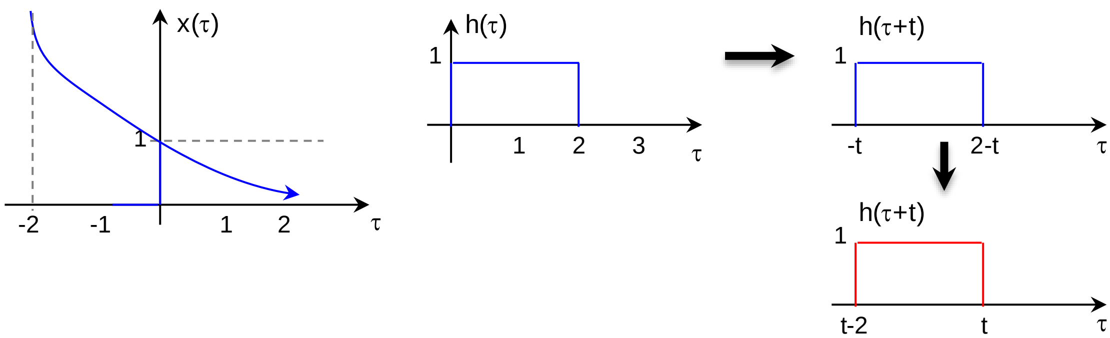
-2 < t < ∞



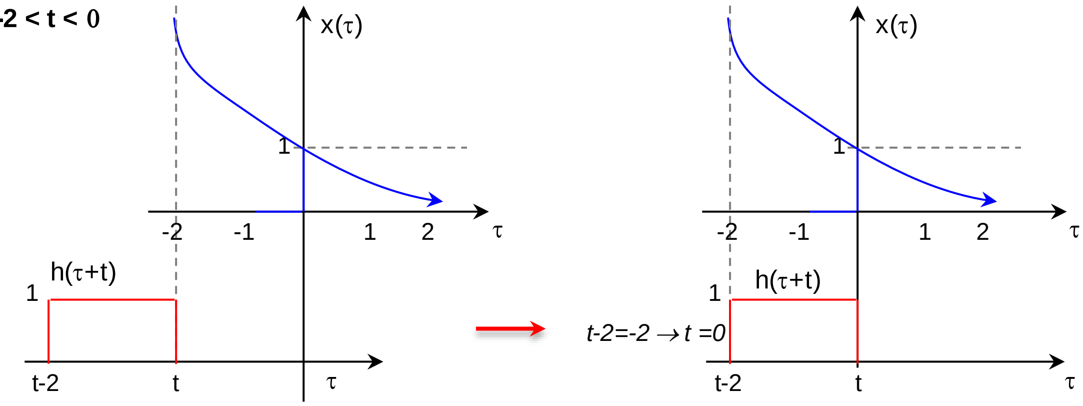
- $-2 < t < \infty$: $w_t(\tau) = e^{-3\tau}, 0 < \tau < t+2, \rightarrow y(t) = \int_{0}^{t+2} e^{-3\tau}d\tau = \frac{1}{3}\left(1 - e^{-3(t+2)}\right), t > -2$
- $t < -2$: $w_t(\tau) = 0, \forall\tau, \rightarrow y(t) = 0, t < -2$

d) $x(t) = e^{-2t}(u(t+2) - u(t-2))$ $h(t) = u(t) - u(t-2)$

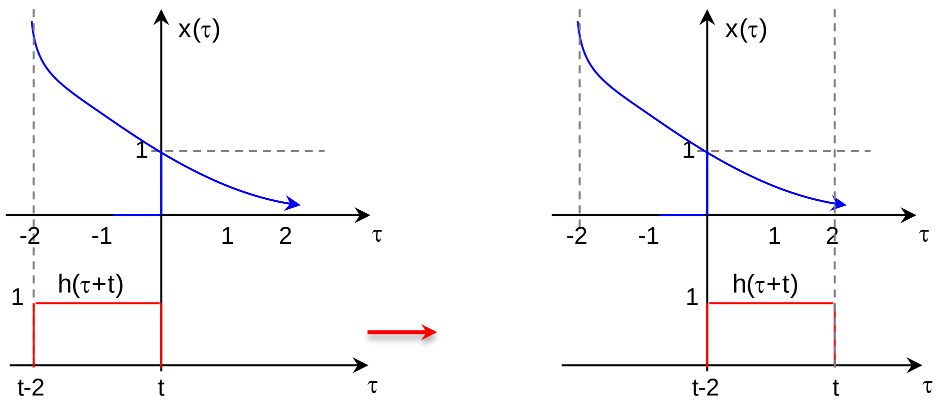
- $-2 < t < 0$: $w_t(\tau) = e^{-2\tau}, -2 < \tau < t, \rightarrow y(t) = \int_{-2}^{t} e^{-2\tau}d\tau = \frac{1}{2}(e^4 - e^{-2t}), -2 < t < 0$
- $0 < t < 2$: $w_t(\tau) = e^{-2\tau}, t-2 < \tau < t, \rightarrow y(t) = \int_{t-2}^{t} e^{-2\tau}d\tau = \frac{1}{2}\left(e^{-2(t-2)} - e^{-2t}\right), 0 < t < 2$
- $2 < t < 4$: $w_t(\tau) = e^{-2\tau}, t-2 < \tau < 2, \rightarrow y(t) = \int_{t-2}^{2} e^{-2\tau}d\tau = \frac{1}{2}\left(e^{-2(t-2)} - e^{-4}\right), 2 < t < 4$
- $t < -2$ or $t > 4$: $w_t(\tau) = 0, \forall\tau, \rightarrow y(t) = 0, t < -2$ or $t > 4$



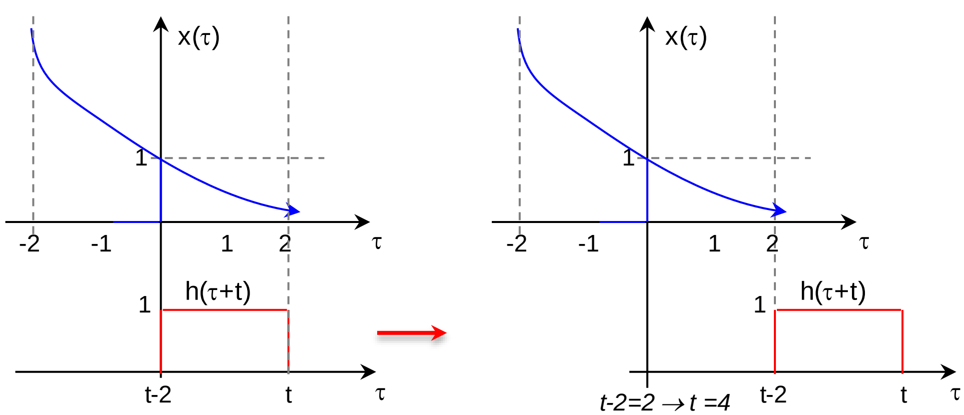
-2 < t < 0



0 < t < 2



2 < t < 4

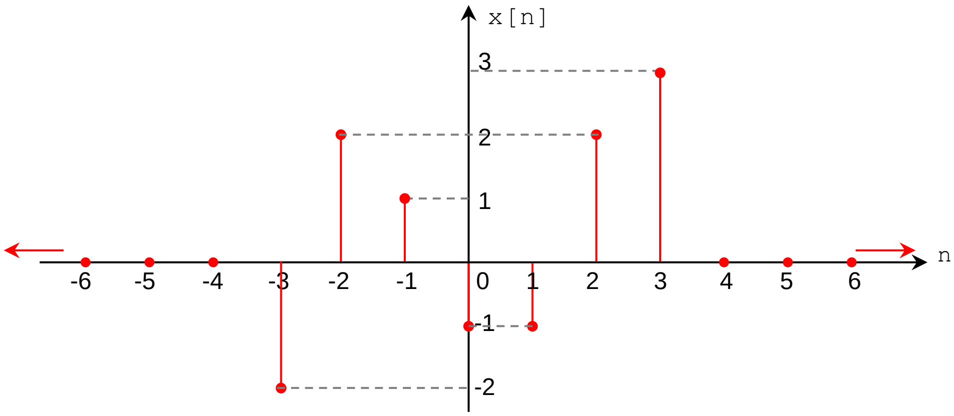


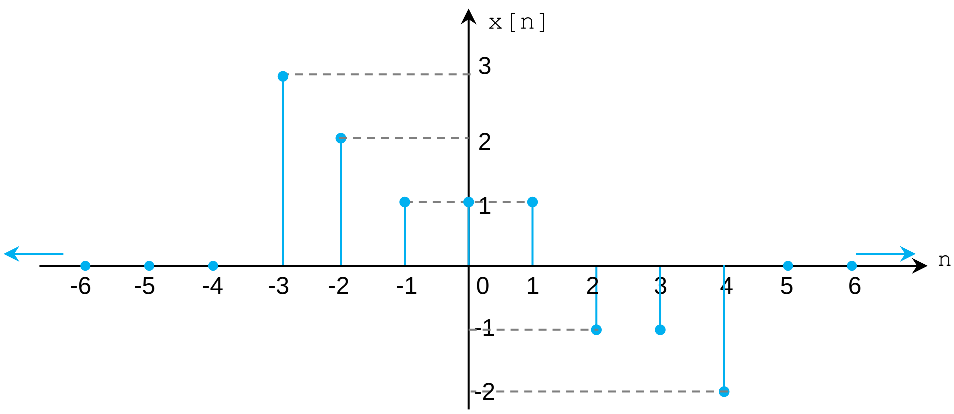
PROBLEM 3

Given the following system: `y[n] = x[n] + 2x[n-1] + 3x[n-2] + 2x[n-3] + x[n-4]`

a) Apply δ`[n]` to the input and obtain the impulse response `h[n]`. Carefully sketch `h[n]`.

b) With the impulse response `h[n]`, you can obtain the output for any input signal `x[n]`. Carefully sketch the output signal `y[n]` for the following input signals. You MUST show the convolution procedure.





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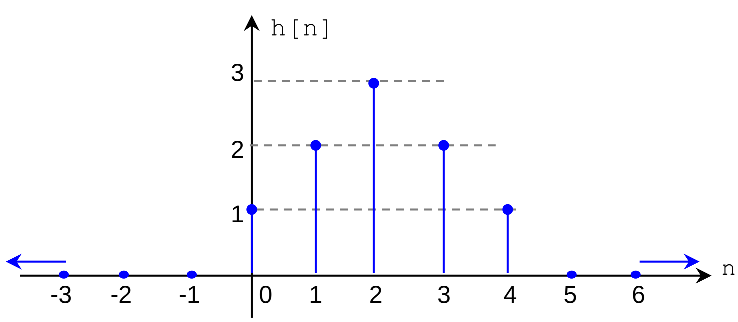
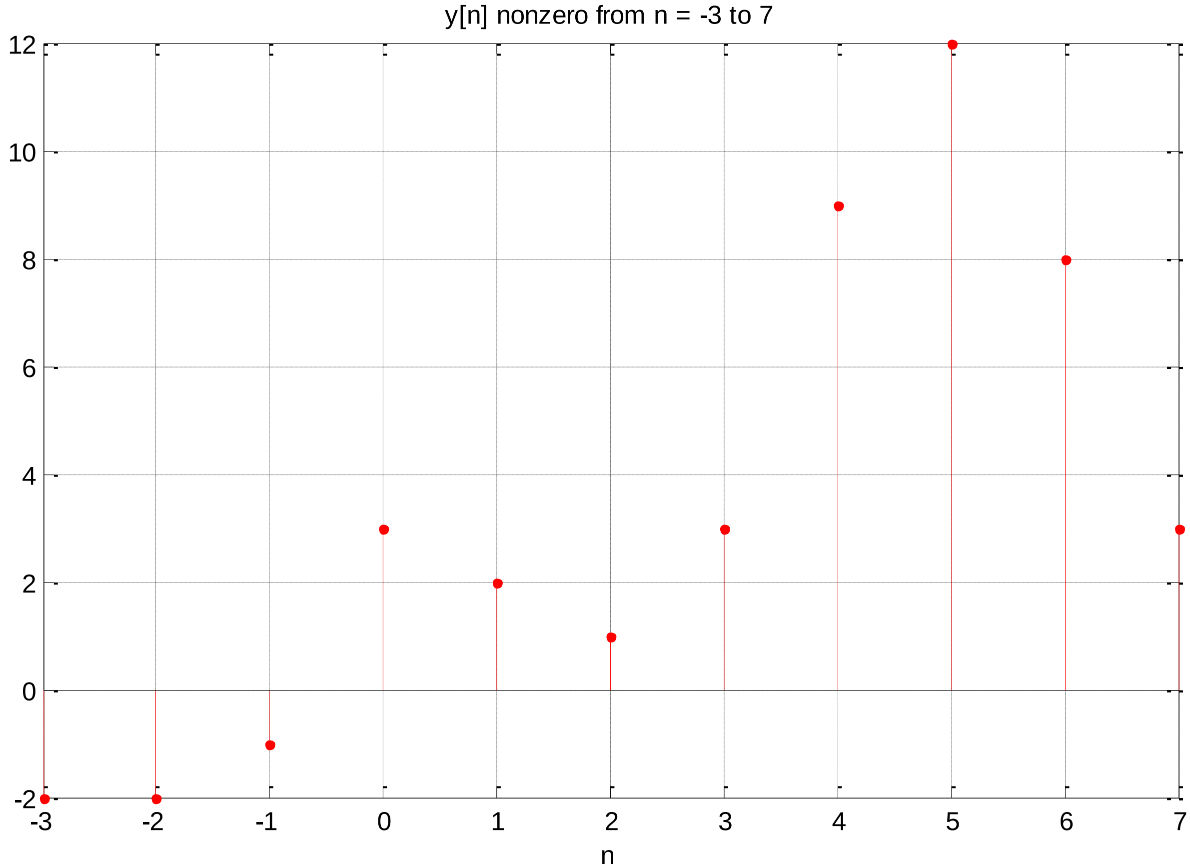
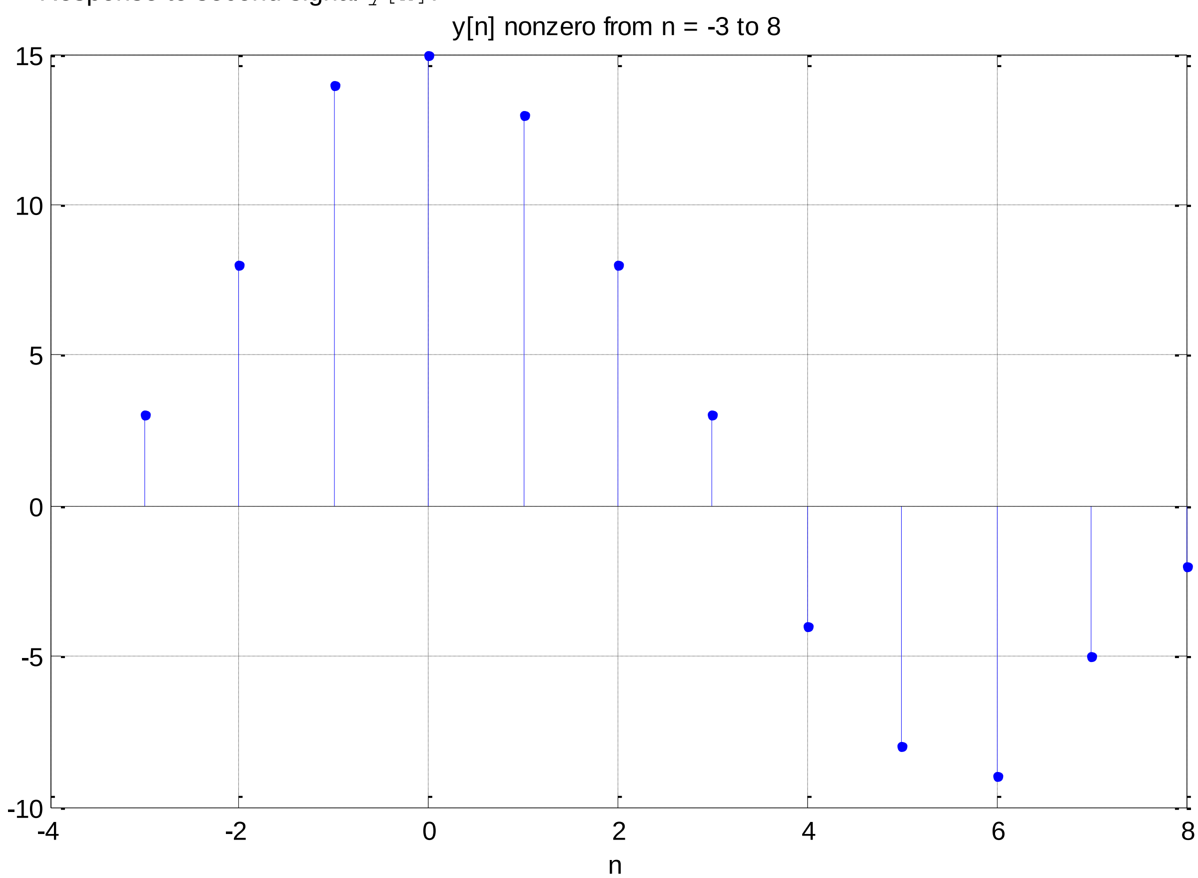
a) `y[n] = `δ`[n] + 2`δ`[n-1] + 3`δ`[n-2] + 2`δ`[n-3] + `δ`[n-4]`



b) Response to first signal `y[n]`:



Response to second signal `y[n]`:



PROBLEM 4

A system (called Moving-Average) has the following input-output relationship:

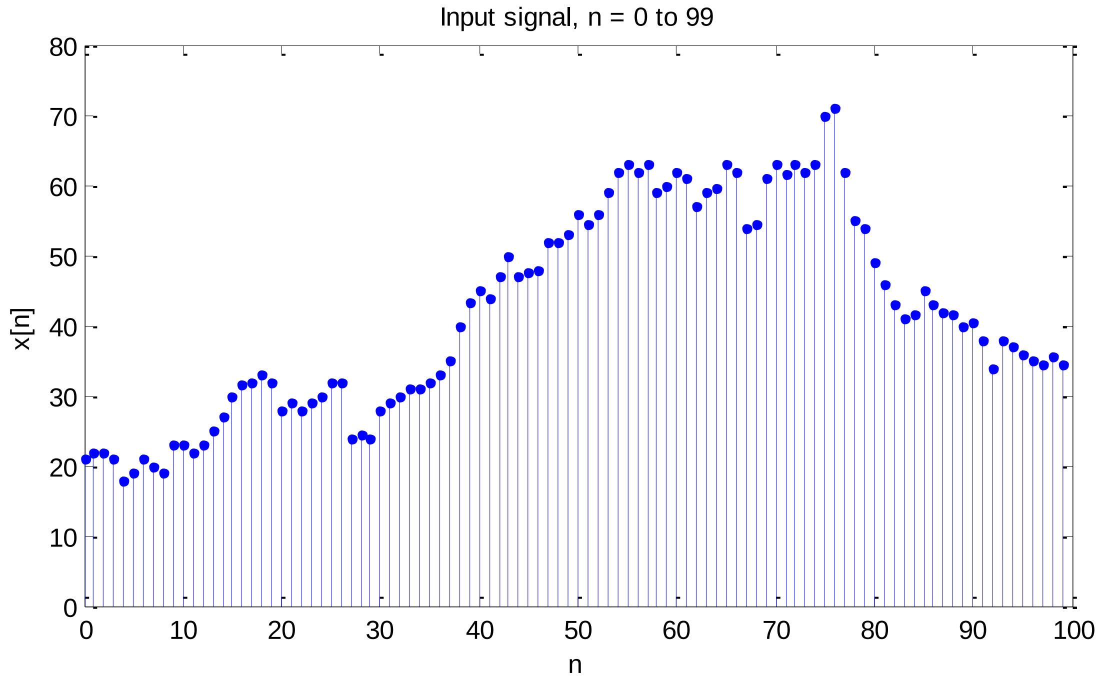
$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

a) Obtain the equation of the impulse response `h[n]`. Sketch `h[n]`.

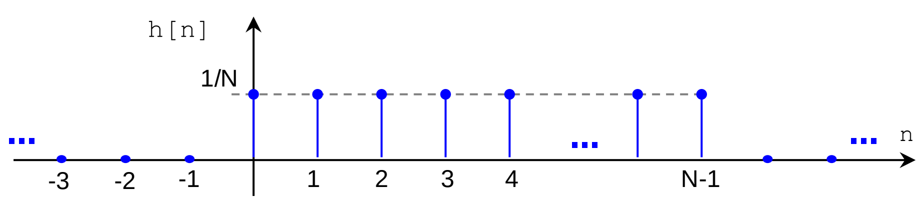
b) Using MATLAB®, plot the response of the system to the input signal below when i) N = 2, ii) N = 5, and iii) N = 10. For each case, explicitly indicate the range of indices for `y[n]`. Attach your MATLAB code to the plots.

c) In your words, explain what effect N has on the shape of the output signal y[n].
Note: The values of the index `n` go from 0 to 99.

```
x = [21 22 22 21 18 19 21 20 19 23 23 22 23 25 27 30 31.5 32 33 32 ...
     28 29 28 29 30 32 32 24 24.5 24 28 29 30 31 31 32 33 35 40 43.2 ...
     45 44 47 50 47 47.5 48 52 52 53 56 54.5 56 59 62 63 62 63 59  ...
     60 62 61 57 59 59.6 63 62 54 54.5 61 63 61.5 63 62 63 70 71 62 ...
     55 54 49 46 43 41 41.5 45 43 42 41.5 40 40.5 38 34 38 37 36 35 ...
     34.5 35.5 34.5];
```



a) $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} \delta[n-k]$



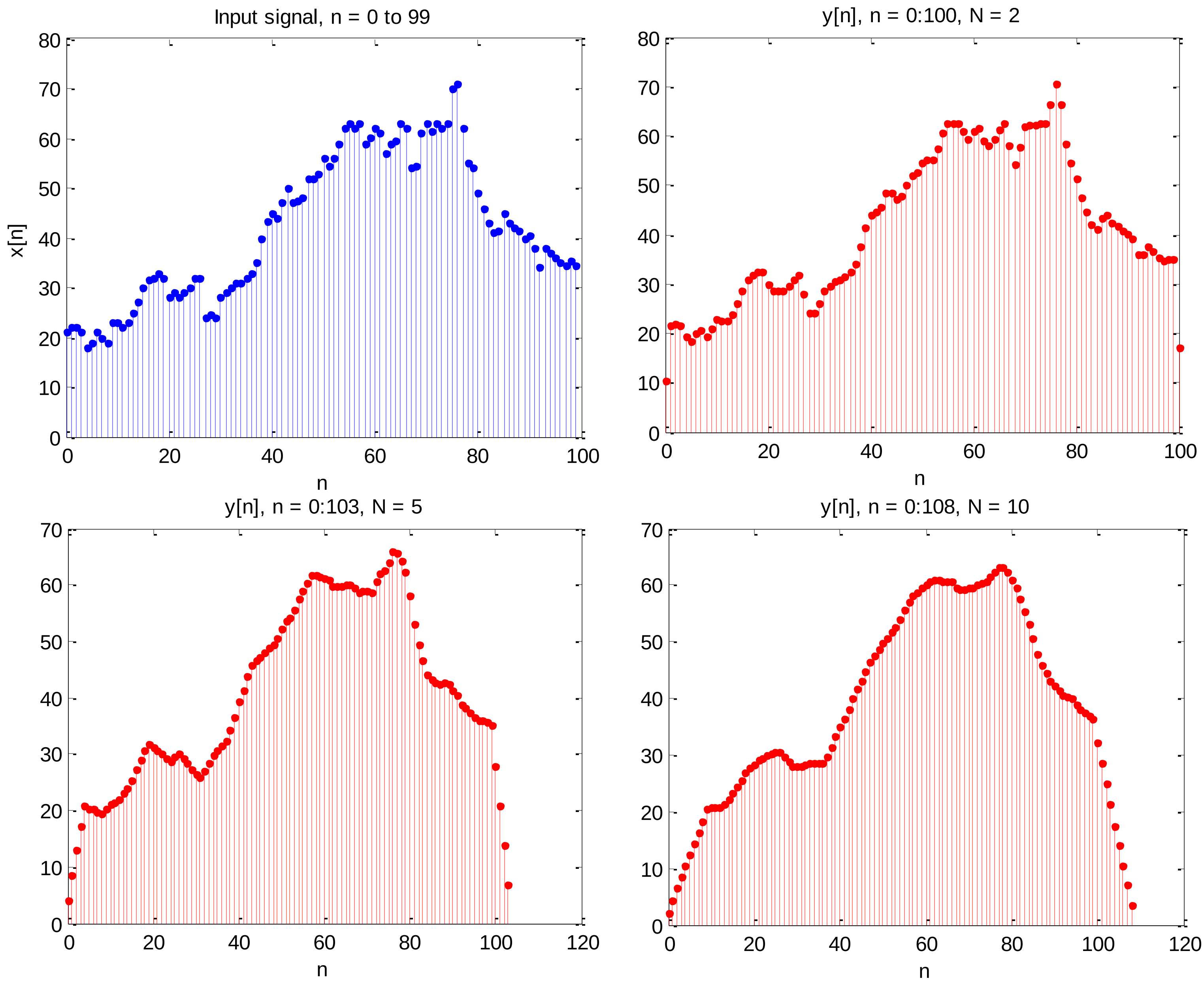
b)

```
clear all; close all; clc
n = 0:99;
x = [21 22 22 21 18 19 21 20 19 23 23 22 23 25 27 30 31.5 32 33 32 ...
     28 29 28 29 30 32 32 24 24.5 24 28 29 30 31 31 32 33 35 40 43.2 ...
     45 44 47 50 47 47.5 48 52 52 53 56 54.5 56 59 62 63 62 63 59  ...
     60 62 61 57 59 59.6 63 62 54 54.5 61 63 61.5 63 62 63 70 71 62 ...
     55 54 49 46 43 41 41.5 45 43 42 41.5 40 40.5 38 34 38 37 36 35 ...
     34.5 35.5 34.5];
```

```
figure; stem(n,x, '.b'); xlabel ('n'); ylabel ('x[n]'); title ('x[n], n = 0 to 99');

N_vec = [2 5 10];
for i = 1:length(N_vec)
    N = N_vec(i);
    h = (1/N)*ones(1,N);
    y{i}= conv(x,h);
    ny{i} = 0:length(x) + length(h) - 1 - 1;
end

figure; stem(ny{1},y{1},'.b'); xlabel('n');
figure; stem(ny{2},y{2},'.b'); xlabel('n');
figure; stem(ny{3},y{3},'.b'); xlabel('n');
```



c) As `N` grows, the output signal `y[n]` gets smoother.

PROBLEM 5

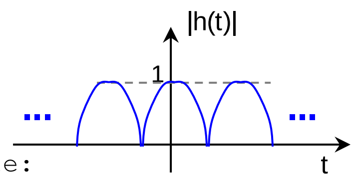
For each of the following impulse responses, determine whether the corresponding LTI system is:
(i) memoryless, (ii) causal, (iii) stable.
Justify your answers.

a) $h(t) = \sin(\pi t)$
b) $h(t) = e^{-3t}u(t-2)$
c) $h(t) = 2\delta(t)$
d) $h[n] = (-1)^n u[-n]$
e) $h[n] = 3u[n-1] - 2u[n-4]$
f) $h[n] = \cos(\pi n)(u[n-2] - u[n+2])$

---

a) $h(t) = \sin(\pi t)$
$h(t) \neq c\delta(t) \rightarrow$ System is NOT memoryless.
$h(t) \neq 0$, for $t < 0 \rightarrow$ System is NOT causal.

$\int_{-\infty}^{\infty}|h(t)|dt$ tends to infinity $\rightarrow$ System is NOT stable:



b) $h(t) = e^{-3t}u(t-2)$
$h(t) \neq c\delta(t) \rightarrow$ System is NOT memoryless.
$h(t) = 0$, for $t < 0 \rightarrow$ System is causal.

$\int_{-\infty}^{\infty}|h(t)|dt = \int_{2}^{\infty}e^{-3t}dt = \frac{e^{-6}}{3} \rightarrow$ System is stable.

c) $h(t) = 2\delta(t)$
$h(t) = c\delta(t) \rightarrow$ System is memoryless.
$h(t) = 0$, for $t < 0 \rightarrow$ System is causal.

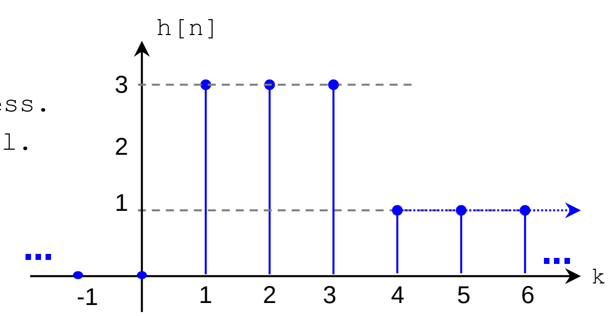
$\int_{-\infty}^{\infty}|h(t)|dt = \int_{-\infty}^{\infty}2\delta(t)dt = 2 \rightarrow$ System is stable.

d) $h[n] = (-1)^n u[-n]$
$h[n] \neq c\delta[n] \rightarrow$ System is NOT memoryless.
$h[n] \neq 0$, for $n < 0 \rightarrow$ System is NOT causal.

$\sum_{-\infty}^{\infty}|h[n]| = \sum_{-\infty}^{\infty}u[-n]$ tends to infinity $\rightarrow$ System is NOT stable.

e) $h[n] = 3u[n-1] - 2u[n-4]$

$h[n] \neq c\delta[n] \rightarrow$ System is NOT memoryless.
$h[n] = 0$, for $n < 0 \rightarrow$ System is causal.



$\sum_{-\infty}^{\infty}|h[n]|$ tends to infinity $\rightarrow$ System is NOT stable.

f) $h[n] = \cos(\pi n)(u[n-2] - u[n+2])$
$h[n] \neq c\delta[n] \rightarrow$ System is NOT memoryless.
$h[n] \neq 0$, for $n < 0 \rightarrow$ System is NOT causal.

$\sum_{-\infty}^{\infty}|h[n]| = \sum_{-2}^{1}|\cos(\pi n)| = 4 \rightarrow$ System is stable.

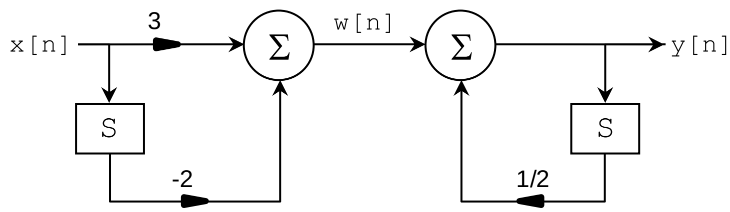
PROBLEM 6

Draw the direct form I and direct form II implementation for the following difference equations:

a) $y[n] - (1/2)y[n-1] = 3x[n] - 2x[n-1]$
b) $y[n] + (1/4)y[n-1] - y[n-3] + (1/2)y[n-4] = x[n-1] - 2x[n-2]$
c) $y[n] - (1/8)y[n-2] = 4x[n-2]$
d) $y[n] - (1/3)y[n-1] = x[n] - x[n-2]$

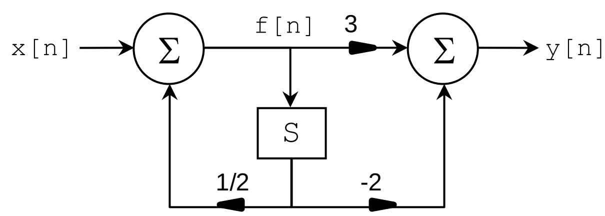
a) $y[n] - (1/2)y[n-1] = 3x[n] - 2x[n-1]$

Direct Form I:



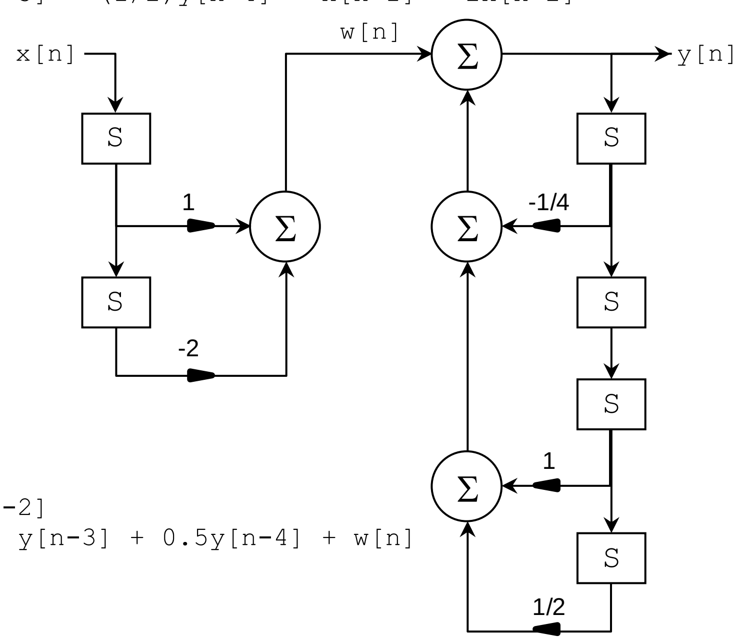
H1: $w[n] = 3x[n] - 2x[n-1]$
H2: $y[n] = 0.5y[n-1] + w[n]$

Direct Form II:
H2: $f[n] = 0.5f[n-1] + x[n]$
H1: $y[n] = 3f[n] - 2f[n-1]$



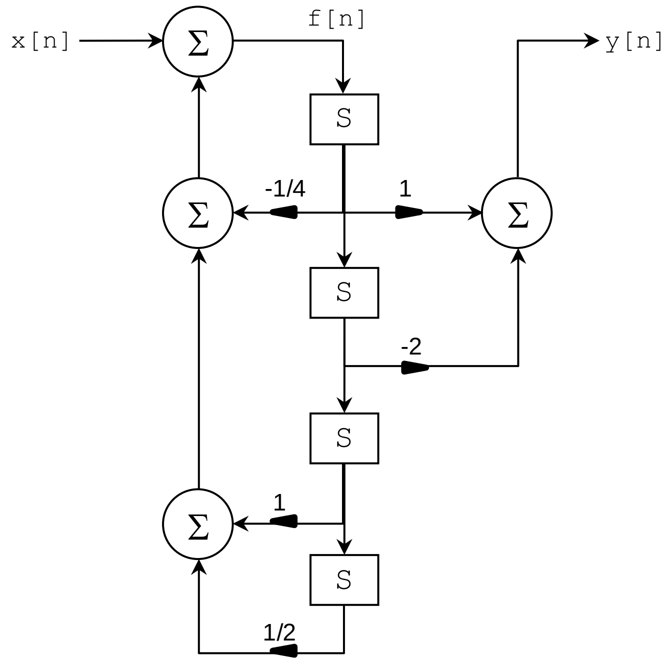
b) $y[n] + (1/4)y[n-1] - y[n-3] + (1/2)y[n-4] = x[n-1] - 2x[n-2]$

Direct Form I:



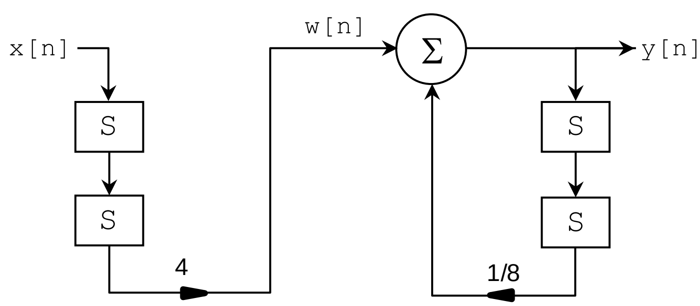
H1: $w[n] = x[n-1] - 2x[n-2]$
H2: $y[n] = -0.25y[n-1] + y[n-3] + 0.5y[n-4] + w[n]$

Direct Form II:
H2: f[n] = -0.25f[n-1] + f[n-3] + 0.5f[n-4] + x[n]
H1: y[n] = f[n-1] - 2f[n-2]

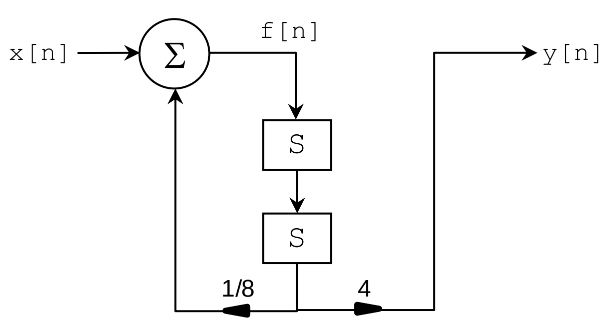


c) y[n] - (1/8)y[n-2] = 4x[n-2]

Direct Form I:
H1: w[n] = 4x[n-2]]
H2: y[n] = 0.125y[n-2] + w[n]

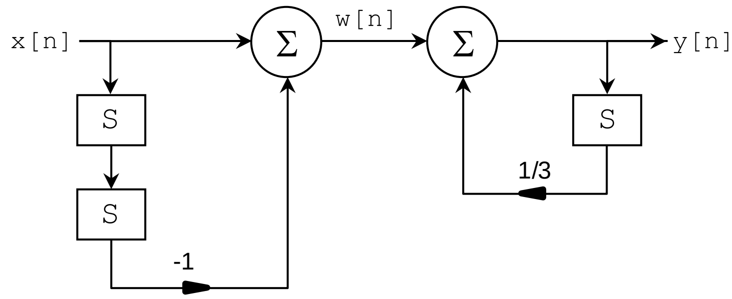


Direct Form II:
H2: f[n] = 0.125f[n-2] + x[n]
H1: y[n] = 4f[n-2]

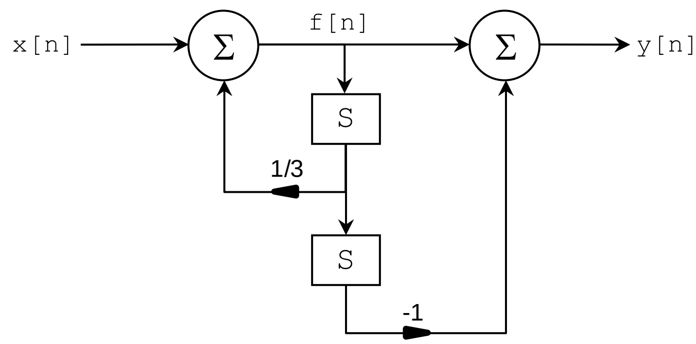


d) y[n] - (1/3)y[n-1] = x[n] - x[n-2]

Direct Form I:
H1: w[n] = x[n] - x[n-2]
H2: y[n] = (1/3)y[n-1] + w[n]

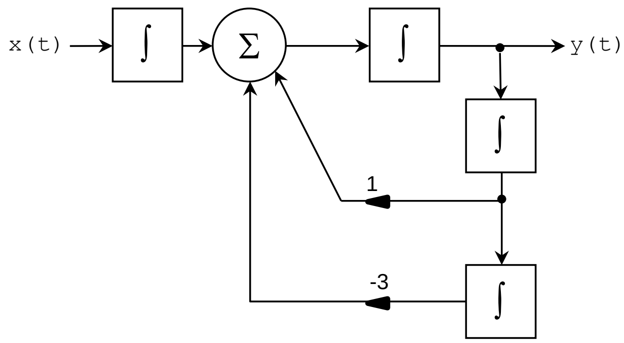


Direct Form II:
H2: f[n] = (1/3)f[n-1] + x[n]
H1: y[n] = f[n] - f[n-2]

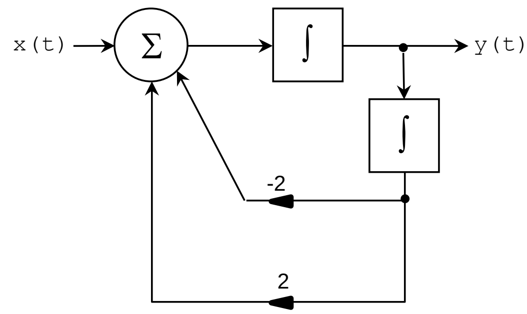


PROBLEM 7

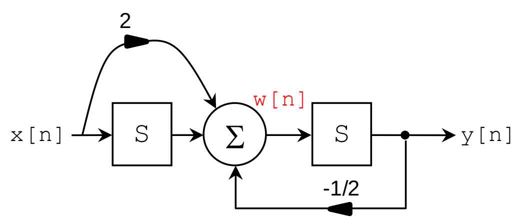
Find the differential-equation or difference-equation description for each of the systems depicted below:



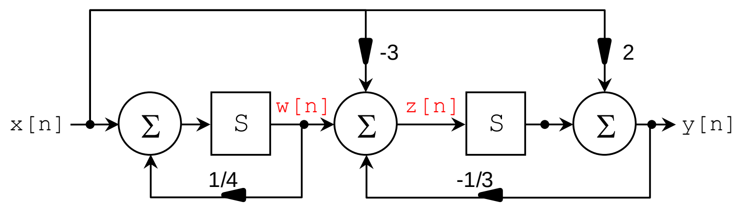
(a)



(b)



(c)



(d)

---

a) $y(t) = \int_{-\infty}^{t} \left(x^{(1)}(\tau) + y^{(1)}(\tau) - 3y^{(2)}(\tau) \right) d\tau$

$$\frac{dy^3}{dt^3} = \frac{dx}{dt} + \frac{dy}{dt} - 3y(t)$$

b) $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

$$\frac{dy}{dt} = x(t)$$

c) w[n] = 2x[n] + x[n-1] - 0.5y[n]
y[n] = w[n-1] = 2x[n-1] + x[n-2] - 0.5y[n-1]

d) w[n] = x[n-1] + 0.25w[n-1]
z[n] = (-1/3)y[n] - 3x[n] + w[n]
y[n] = 2x[n] + z[n-1]
y[n] = 2x[n] -(1/3)y[n-1] - 3x[n-1] + w[n]