Solutions - Homework # 1

Problem 1
A continuous time signal is shown in the figure. Carefully sketch each of the following signals:

a) $x(t-3)$
b) $x(2-t)$
c) $x(2t+2)$
d) $x(2 - \frac{t}{3})$
e) $x(t) \ast (\delta(t+3/2) - \delta(t-3/2))$
f) $(x(t) + x(2-t)) \ast u(1-t)$
g) $x(t/2 - 3) + x(t/3 - 2)$
h) $x(t-1) \ast u(t-1)$

---

Graphs of each signal are shown below:

- **a)** $x(t-3)$
- **b)** $x(2-t)$
- **c)** $x(2t+2)$
- **d)** $x(2 - \frac{t}{3})$
- **e)** $x(t) \ast (\delta(t+3/2) - \delta(t-3/2))$
- **f)** $(x(t) + x(2-t)) \ast u(1-t)$
- **g)** $x(t/2 - 3) + x(t/3 - 2)$
- **h)** $x(t-1) \ast u(t-1)$
\[ x(t) = \begin{cases} 2 & \text{for } 1 \leq t \leq 2 \\ 1 & \text{for } 0 \leq t < 1 \\ -2 & \text{for } -1 \leq t < 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ x(t-1) = \begin{cases} 2 & \text{for } 3 \leq t \leq 4 \\ 1 & \text{for } 2 \leq t < 3 \\ -1 & \text{for } 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ \delta(t+3/2) - \delta(t-3/2) \]

\[ x(2-t) \]

\[ u(1-t) \]

\[ (x(t) + x(2-t))u(1-t) \]
\[ x(t/2 - 3) + x(t/3 - 2) \]
PROBLEM 2

The discrete-time signals $x_1[n]$ and $x_2[n]$ are shown in the figure. Carefully sketch each of the following signals:

- **a)** $x_1[3n] + x_2[n-2]$
- **b)** $x_1[2n] - u[n-2]$
- **c)** $0.5x_1[n] + (-1)^n x_1[n]$
- **d)** $x_1[n]*u[1-n]$
- **e)** $x_1[2n-3]$
- **f)** $x_1[2n-1] + u[2n+3]$
- **g)** $x_1[n-1]\delta[n-3]$
- **h)** $2^n x_1[n-1]$

![Graph showing the sketches of the signals](image-url)
c) $0.5x_1[n]$

$(-1)^n x_1[n]$

$0.5x_1[n] + (-1)^n x_1[n]$

e) $x_1[n-3]$

f) $x_1[2n-3]$
PROBLEM 3

Determine whether the following signals are periodic, and for those which are, find the fundamental period (T for continuous time signals and N for discrete-time signals) and the fundamental angular frequencies (ω for continuous time signals and Ω for discrete-time signals). You must specify the units of these quantities.

a) \( x[n] = \cos((8/15)\pi n) \)
b) \( x[n] = \sin((7/15)\pi n) \)
c) \( x(t) = \sin(2t) + \cos(3t) \)
d) \( x[n] = \sin((1/5)\pi n)\sin((1/3)\pi n) \)
e) \( x(t) = \sin(t)u(t-1) \)
f) \( x(t) = \sin(t)u(t) + \sin(-t)u(-t) \)

a) \( N = 2\pi m/(8/15\pi) \rightarrow N = (15/4)m \)
Then, we choose \( m = 4 \rightarrow N = 15 \) samples, \( \Omega = 2\pi/15 \) rads/sample.

b) \( N = 2\pi m/(7/15\pi) \rightarrow N = (30/7)m \)
Then, we choose \( m = 7 \rightarrow N = 30 \) samples, \( \Omega = \pi/15 \) rads/sample.

c) For periodicity: \( x(t)=x(t+T) \)
\( \sin(2t) + \cos(3t) = \sin(2t + 2T) + \cos(3t + 3T) \)
We need: \( 2T = 2\pi k, \text{ and } 3T = 2\pi r, \text{ where } k,r \text{ are integers} \)
Then, we see that: \( T = \pi k = (2\pi/3)r \rightarrow 3k = 2r \)
The smallest numbers \( k,r \) that satisfy this condition are \( k = 2, \ r = 3. \)

Then, the signal \( x(t) \) is periodic with period \( T = 2\pi \) secs, \( \omega = 1 \) rad per second

d) For periodicity: \( x[n] = x[n + N] \)
\( \sin(\pi n/5)\sin(\pi n/3) = \sin(\pi n/5 + \pi N/5)\sin(\pi n/3 + \pi N/3) \)
We need: \( \pi N/5 = 2\pi k, \text{ and } \pi N/3 = 2\pi r, \text{ where } k,r \text{ are integers} \)
Then, we see that: \( N = 10k = 6r \rightarrow 5k = 3r \)
The smallest numbers \( k,r \) that satisfy this condition are \( k = 3, \ r = 5. \)

Then, the signal \( x[n] \) is periodic with period \( N = 30 \) samples, \( \Omega = \pi/15 \) rads/sample.

* Optional:
If we notice that: \( \sin(\pi n/5)\sin(\pi n/3) = \sin(\pi n/5 + \pi k)\sin(\pi n/3 + \pi r) \)
Then, we need: \( \pi N/5 = \pi k, \text{ and } \pi N/3 = \pi r, \text{ where } k,r \text{ are integers} \)
Then, \( k=3, \ r=5, \text{ and } N = 5k = 3r = 15 \) samples, \( \Omega = 2\pi/15 \) rads/sample.

e) The signal is zero only for \( t \leq 1. \) Therefore, \( x(t) \) is non-periodic.

f) The signal is non-periodic: \( x(t) \neq x(t + T) \)
PROBLEM 4
The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$ respectively. For each system, determine (and justify) whether it is (i) memoryless, (ii) stable, (iii) causal, (iv) linear, and (v) time invariant. Recall that to disprove that a system has a certain property, all you need is to come up with a counter-example.

a) $y(t) = \cos(x(t))$

It is memoryless: it only depends on the current value of the signal.

Stability:
If $|x(t)| \leq Mx < \infty, \forall t \rightarrow$ it should be that $|y(t)| \leq My < \infty, \forall t$
$|y(t)| = |\cos(x(t))| \rightarrow |y(t)| = |\cos(x(t))| \leq 1$

Therefore, the system is stable.

Causality: It is causal, because it does not depend on future values of the input $x(t)$.

Linearity:
If the input to the system is $ax_A(t) + bx_B(t)$, where $a, b$, are real numbers, then the output to the system should be $ay_A(t) + by_B(t)$, where $y_A(t), y_B(t)$ are the responses of the system to $x_A(t)$ and $x_B(t)$ respectively.

If the input to the system is $ax_A(t) + bx_B(t)$, then:
$y(t) = \cos(ax_A(t) + bx_B(t))$,
$ay_A(t) + by_B(t) = \cos(x_A(t)) + b\cos(x_B(t))$

We see that $y(t) \neq a(y_A(t)) + b(y_B(t)).$ Thus, the system is NOT linear.

Time invariance:
We have a system $y(t) = H(x(t))$. The response of the system to a shifted input $x(t-k)$ should be the same as if the output $y(t)$ has been shifted by $k$, i.e., $y(t-k)$:

Response of system to shifted input $x(t-k): y_1(t) = \cos(x(t-k))$

Output $y(t)$ shifted by $k$: $y(t-k) = \cos(x(t-k))$

We see that $y(t-k) = y_1(t)$. Thus, the system is time invariant.
b) \( y(t) = x(t/2) \)

It is **NOT** memoryless: \( y(2) = x(1), y(3) = x(1.5) \).

**Stability:**
If \( |x(t)| \leq M_x < \infty, \forall t \rightarrow \) it should be that \( |y(t)| \leq M_y < \infty, \forall t \)

\[ |y(t)| = |x(t/2)| \leq |x(t)| \]

Therefore, the system is stable.

**Causality:** It is **NOT** causal, because it depends on future values of the input \( x(t) \):
For example: \( y(-1) = x(-0.5), y(-4) = x(-2) \).

**Linearity:**
If the input to the system is \( ax_A(t) + bx_B(t) \), where \( a, b \) are real numbers, then the output to the system should be \( ay_A(t) + by_B(t) \), where \( y_A(t), y_B(t) \) are the responses of the system to \( x_A(t) \) and \( x_B(t) \) respectively.

If the input to the system is \( ax_A(t) + bx_B(t) \), then:
\[
ay_A(t) + by_B(t) = ax_A(t/2) + bx_B(t/2)
\]

We see that \( y(t) = ay_A(t) + by_B(t) \). Thus, the system is linear.

**Time invariance:**
We have a system \( y(t) = H(x(t)) = x(t/2) \). The response of the system to a shifted input \( x(t-k) \) should be the same as if the output \( y(t) \) has been shifted by \( k \), i.e., \( y(t-k) \).

\[
x(t) \xrightarrow{S^k} (t-k) \xrightarrow{H} y_1(t) = x(t/2-k)
\]

\[
x(t) \xrightarrow{H} y(t) = x(t/2) \xrightarrow{S^k} y(t-k) = x((t-k)/2)
\]

Response of system to shifted input \( x(t-k) \): \( y_1(t) = x(t/2-k) \)

Output \( y(t) \) shifted by \( k \): \( y(t-k) = x((t-k)/2) \)

We see that \( y(t-k) \neq y_1(t) \). Thus, the system is **NOT** time invariant.

c) \( y[n] = 3x[n]u[n] \)

It is **memoryless**: it only depends on the current sample of the signal.

**Stability:**
If \( |x[n]| \leq M_x < \infty, \forall n \rightarrow \) it should be that \( |y[n]| \leq M_y < \infty, \forall n \)

\[ |y[n]| = |3x[n]u[n]| \leq |3x[n]| \]

Therefore, the system is stable.

**Causality:** It is causal, because it does not depend on future samples of the input \( x[n] \).

**Linearity:**
If the input to the system is \( ax_A[n] + bx_B[n] \), where \( a, b \) are real numbers, then the output to the system should be \( ay_A[n] + by_B[n] \), where \( y_A[n], y_B[n] \) are the responses of the system to \( x_A[n] \) and \( x_B[n] \) respectively.
If the input to the system is $ax_A[n] + bx_B[n]$, then:

\[
y[n] = 3(ax_A[n] + bx_B[n])u[n] \\
y[n] = 3(ax_A[n]u[n]) + 3(bx_B[n]u[n])
\]

\[
ay_A[n] + by_B[n] = a*3*x_A[n]u[n] + b*3*x_B[n]
\]

We see that $y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

**Time invariance:**
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by $k$, i.e., $y[n-k]$:

\[
x[n] \rightarrow S^k \rightarrow x[n-k] \rightarrow H \rightarrow y_1[n] = 3x[n-k]u[n]
\]

\[
x[n] \rightarrow H \rightarrow y[n]=3x[n]u[n] \rightarrow S^k \rightarrow y[n-k]=3x[n-k]u[n-k]
\]

Response of system to a shifted input $x[n-k]$: $y_1[n] = 3x[n-k]u[n]$  
Output $y[n]$ shifted by $k$: $y[n-k] = 3x[n-k]u[n-k]$

We see that $y[n-k] \neq y_1[n]$. Thus, the system is NOT time invariant.

d) $y[n] = \log_2(|x[n]|)$

It is **memoryless**: it only depends on the current sample of the signal.

**Stability:**
If $|x[n]| \leq Mx < \infty$, $\forall n \rightarrow$ it should be that $|y[n]| \leq My < \infty$, $\forall n$

\[
|y[n]| = |\log_2|x[n]||
\]

When $x[n]$ approaches 0, $\log_2(|x[n]|)$ approaches towards minus infinity. Thus, the system is NOT stable.

**Causality:** It is causal, because it does not depend on future samples of the input $x[n]$.

**Linearity:**
If the input to the system is $ax_A[n] + bx_B[n]$, where $a, b$ are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n], y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $ax_A[n] + bx_B[n]$, then:

\[
y[n] = \log_2(|ax_A[n] + bx_B[n]|)
\]

\[
ay_A[n] + by_B[n] = a\log_2(|x_A[n]|) + b\log_2(|x_B[n]|)
\]

We see that $y[n] \neq a(y_A[n]) + b(y_B[n])$. Thus, the system is NOT linear.

**Time invariance:**
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by $k$, i.e., $y[n-k]$:

Response of system to a shifted input $x[n-k]$: $y_1[n] = \log_2(|x[n-k]|)$

Output $y[n]$ shifted by $k$: $y[n-k] = \log_2(|x[n-k]|)$

We see that $y[n-k] = y_1[n]$. Thus, the system is time invariant.
e) \( y[n] = x[n] + x[n-1] + x[n+2] \)

It is NOT memoryless: it depends on previous and future samples of the signal.

**Stability:**

If \( |x[n]| \leq Mx < \infty, \forall n \rightarrow \) it should be that \( |y[n]| \leq My < \infty, \forall n \)

\( |y[n]| = |x[n] + x[n-1] + x[n+2]| \leq |x[n]| + |x[n-1]| + |x[n+2]| \)

If \( |x[n]| \leq Mx < \infty, \forall n \rightarrow |x[n-k]| \leq Mx < \infty, \forall n \)

Then:

\( |y[n]| \leq Mx + Mx + Mx \)

\( |y[n]| \leq 3Mx \)

Therefore, the system is stable.

**Causality:** It is NOT causal, because it depends on future samples of the input \( x[n] \).

**Linearity:**

If the input to the system is \( ax_A[n] + bx_B[n] \), where \( a, b \), are real numbers, then the output to the system should be \( ay_A[n] + by_B[n] \), where \( y_A[n], y_B[n] \) are the responses of the system to \( x_A[n] \) and \( x_B[n] \) respectively.

If the input to the system is \( ax_A[n] + bx_B[n] \), then:

\( y[n] = (ax_A[n] + bx_B[n]) + (ax_A[n-1] + bx_B[n-1]) + (ax_A[n+2] + bx_B[n+2]) \)

\( y[n] = a(x_A[n] + x_A[n-1] + x_A[n+2]) + b(x_B[n] + x_B[n-1] + x_B[n+2]) \)

\( \rightarrow y[n] = a(y_A[n]) + b(y_B[n]) \). Thus, the system is linear.

**Time invariance:**

We have a system \( y[n] = H(x[n]) \). The response of the system to a shifted input \( x[n-k] \) should be the same as if the output \( y[n] \) has been shifted by \( k \), i.e., \( y[n-k] \):

Response of system to a shifted input \( x[n-k] \): \( y_1[n] = x[n-k] + x[n-k-1] + x[n-k+2] \)

Output \( y[n] \) shifted by \( k \): \( y[n-k] = x[n-k] + x[n-k-1] + x[n-k+2] \)

We see that \( y[n-k] = y_1[n] \). Thus, the system is time invariant.

f) \( y[n] = 2^n x[n] \)

It is memoryless: it only depends on the current sample of the signal.

**Stability:**

If \( |x[n]| \leq Mx < \infty, \forall n \rightarrow \) it should be that \( |y[n]| \leq My < \infty, \forall n \)

\( |y[n]| = |2^n x[n]| \leq 2^n |x[n]| \)

As \( n \) grows, \( 2^n \) tends to infinity. Therefore, the system is NOT stable.

**Causality:** It is causal, because it does not depend on future samples of the input \( x[n] \).

**Linearity:**

If the input to the system is \( ax_A[n] + bx_B[n] \), where \( a, b \), are real numbers, then the output to the system should be \( ay_A[n] + by_B[n] \), where \( y_A[n], y_B[n] \) are the responses of the system to \( x_A[n] \) and \( x_B[n] \) respectively.

If the input to the system is \( ax_A[n] + bx_B[n] \), then:
\[ y[n] = 2^n(ax_a[n] + bx_b[n]) \]
\[ y[n] = 2^n(ax_a[n]) + 2^n(bx_b[n]) \]
\[ ay_a[n] + by_b[n] = a2^nax_a[n]u[n] + b2^nx_b[n] \]

We see that \( y[n] = a(y_a[n]) + b(y_b[n]) \). Thus, the system is linear.

**Time invariance:**

We have a system \( y[n] = H(x[n]) \). The response of the system to a shifted input \( x[n-k] \) should be the same as if the output \( y[n] \) has been shifted by \( k \), i.e., \( y[n-k] \):

\[
\begin{align*}
x[n] & \rightarrow S^k \rightarrow x[n-k] \\
& \rightarrow H \rightarrow y_1[n] = 2^n x[n-k] \\
\end{align*}
\]

\[
\begin{align*}
x[n] & \rightarrow H \rightarrow y[n] = 2^n x[n] \\
& \rightarrow S^k \rightarrow y[n-k] = 2^n x[n-k] \\
\end{align*}
\]

Response of system to a shifted input \( x[n-k] \):
\[ y_1[n] = 2^n x[n-k] \]

Output \( y[n] \) shifted by \( k \):
\[ y[n-k] = 2^n x[n-k] \]

We see that \( y[n-k] \neq y_1[n] \). Thus, the system is NOT time invariant.

**Problem 5**

Using MATLAB®, plot (with the command 'stem') the following signals for \( n = -40 \) to 40. Attach your MATLAB code to the plots.

a) \( x[n] = 0.6(0.95)^n \)
b) \( x[n] = \cos((\pi/12)\,n + \pi/3) + \sin((\pi/6)\,n + \pi/5) \)
c) \( x[n] = A\cos(\Omega_0\,n + \phi) \) for:
   i. \( A = 2.5, \Omega_0 = 2\pi/45, \phi = \pi/5 \)
   ii. \( A = 0.5, \Omega_0 = \pi/12, \phi = \pi/3 \)
   iii. \( A = 1.5, \Omega_0 = \pi/2, \phi = \pi/5 \)

```matlab
clear all; close all; clc;
% Generating vector of samples:
n = -40:40; % [-40 -19 ... 40]

% a) x[n] = 0.6*(0.95)^n
B = 0.6; r = 0.95;
x1 = B*(r.^n);
figure; stem (n,x1,'.r'); title ('r = 0.95'); xlabel ('n');

% b) x[n] = \cos((\pi/12)*n + \pi/3) + \sin((\pi/6)*n + \pi/5)
x2 = cos(pi*n/12 + pi/3) + sin(pi*n/6 + pi/5);
figure; stem (n,x2,'.k'); title ('sum of sinusoids'); xlabel ('n');

% c) x[n] = A*cos(omega0*n + phi):
A = 2.5; omega0 = 2*pi/45; phi = pi/5;
x = A*cos(omega0*n + phi);
figure; stem (n,x,'.b'); title ('Sinusoid. A=2.5, omega0=2*pi/45, phi= \pi/5\'); xlabel ('n');
figure; stem (n,x,'.b'); title ('Sinusoid. A=0.5, omega0=\pi/2, phi= \pi/3\'); xlabel ('n');
figure; stem (n,x,'.b'); title ('Sinusoid. A=1.5, omega0 = \pi/12, phi = \pi/5\'); xlabel ('n');
```

Instructor: Daniel Llamocca
a) Sum of sinusoids

b) Sinusoid. $A = 2.5$, $\omega_0 = \frac{2\pi}{45}$, $\phi = \frac{\pi}{5}$

c)i Sinusoid. $A = 0.5$, $\omega_0 = \frac{\pi}{12}$, $\phi = \frac{\pi}{3}$

c)ii Sinusoid. $A = 1.5$, $\omega_0 = \frac{\pi}{2}$, $\phi = \frac{\pi}{5}$
**Problem 6**

Let \( x(t) \) be the continuous-time complex exponential signal:
\[
x(t) = \exp(j\omega_0 t)
\]
with fundamental frequency \( \omega = \omega_0 \), and fundamental period \( T_0 = 2\pi/\omega_0 \).

The discrete-time signal \( x[n] \) was generated by uniformly sampling (taking equally spaced samples) the signal \( x(t) \) with a sampling period \( T_s \) (in seconds)
\[
x[n] = x(nT_s) = \exp(j\omega_0 nT_s)
\]

a. Show that \( x[n] \) is periodic if and only if \( T_s/T_0 \) is a rational number.

b. If \( \omega_0 = \pi/8 \), \( N = 40 \), what is the minimum number of cycles of the original complex exponential (also called envelope cycles) that are required for \( x[n] \) to be periodic?

c. Once you obtained the minimum number of envelope cycles, what is the sampling period (in seconds)?

---

**Problem 7**

Using MATLAB®, plot (with the 'stem' command) the following exponentially damped sinusoidal signal for two different values of \( r \) (one positive and one negative).
\[
x[n] = B r^n \sin(\Omega_0 n + \phi)
\]

Note that \( 0 < |r| < 1 \) (otherwise there is no exponential decay).

Range: \( n = -50 \) to \( 50 \). Fixed parameters: \( \phi = \pi/4 \), \( B = 2 \).

Pick \( \Omega_0 \) and \( r \) judiciously so that a clear damping on the sinusoid can be seen in the plot. Attach your MATLAB code to the plots.

---

```matlab
clear all; close all; clc;
n = -50:50;
B = 2;
phi = pi/4;
omega0 = pi/6;
r = 0.95; % 0 < r < 1: decaying exponential
x = B*(r.^n).*sin(omega0*n + phi);
figure; stem (n,x,'.b'); title ('r = 0.95'); xlabel ('n');
z = B*((-r).^n).*sin(omega0*n + phi);
figure; stem (n,z,'.b'); title ('r = 0.95'); xlabel ('n');
```

---

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PROBLEM 8
The output of a discrete-time system is related to its input \( x[n] \) as follows:
\[
y[n] = a_0 x[n] + a_1 x[n+1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]
\]
where \( a_0, a_1, a_2, a_3, a_4 \) are real values.

Let the operator \( S^k \) denote a system that shifts the input \( x[n] \) by \( k \) samples to produce \( x[n-k] \).

a. Formulate the operator \( H \) for the system relating \( y[n] \) to \( x[n] \). Then develop a block diagram representation for \( H \), using (i) cascade implementation, and (ii) parallel implementation.
b. Demonstrate that the system is BIBO stable for all \( a_0, a_1, a_2, a_3, a_4 \) (real values)
c. Under what condition (if any) of the values \( a_0, a_1, a_2, a_3, a_4 \) is the system causal?
d. Demonstrate that the system is linear and time-invariant.

\[
\rightarrow H = a_0 S^0 + a_1 S^{-1} + a_2 S^2 + a_3 S^3 + a_4 S^4.
\]

\[
\begin{array}{cccccc}
\text{CASCADE} & \\
\begin{array}{c}
 S^{-1} \\
a_1 \\
 S \\
a_0 \\
 S \\
a_2 \\
 S \\
a_3 \\
 S \\
a_4 \\
 \Sigma \\
y[n]
\end{array} & \\
\begin{array}{c}
x[n] \\
\end{array}
\end{array}
\]

\[
\begin{array}{cccccc}
\text{PARALLEL} & \\
\begin{array}{c}
 S^{-1} \\
a_1 \\
 S^2 \\
a_2 \\
 S^3 \\
a_3 \\
 S^4 \\
a_4 \\
 \Sigma \\
y[n] \\
x[n]
\end{array} & \\
\end{array}
\]
b. Stability:
   If \(|x[n]| \leq Mx < \infty, \forall n \rightarrow \) it should be that \(|y[n]| \leq My < \infty, \forall n\)

\[ |y[n]| = |a_0 x[n] + a_1 x[n+1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]| \]
\[ |y[n]| \leq |a_0 x[n]| + |a_1 x[n+1]| + |a_2 x[n-2]| + |a_3 x[n-3]| + |a_4 x[n-4]| \]
\[ |y[n]| \leq |a_0||x[n]| + |a_1||x[n+1]| + |a_2||x[n-2]| + |a_3||x[n-3]| + |a_4||x[n-4]| \]

If \(|x[n]| \leq Mx < \infty, \forall n \rightarrow |x[n-k]| \leq Mx < \infty, \forall n\)
Then:
\[ |y[n]| \leq |a_0|Mx + |a_1|Mx + |a_2|Mx + |a_3|Mx + |a_4|Mx \]
\[ |y[n]| \leq (|a_0| + |a_1| + |a_2| + |a_3| + |a_4|)Mx \]

Therefore, \(y[n]\) is stable.

c. The term \(x[n+1]\) makes the system noncausal. Then, for the system to be causal, we require that \(a_1 = 0\).

d. Linearity:
   If the input to the system is \(a x_A[n] + b x_B[n]\), where \(a, b\), are real numbers, then the output to the system should be \(a y_A[n] + b y_B[n]\), where \(y_A[n]\), \(y_B[n]\) are the responses of the system to \(x_A[n]\) and \(x_B[n]\) respectively.

If the input to the system is \(a x_A[n] + b x_B[n]\), then:
\[ y[n] = a_0 (a x_A[n] + b x_B[n]) + a_1 (a x_A[n+1] + b x_B[n+1]) + a_2 (a x_A[n-2] + b x_B[n-2]) + a_3 (a x_A[n-3] + b x_B[n-3]) + a_4 (a x_A[n-4] + b x_B[n-4]) \]
\[ y[n] = a (a_0 x_A[n] + a_1 x_A[n+1] + a_2 x_A[n-2] + a_3 x_A[n-3] + a_4 x_A[n-4]) + b (a_0 x_B[n] + a_1 x_B[n+1] + a_2 x_B[n-2] + a_3 x_B[n-3] + a_4 x_B[n-4]) \]
\[ \rightarrow y[n] = a (y_A[n]) + b (y_B[n]). \] Thus, the system is linear.

Time invariance:
   We have a system \(y[n] = H(x[n])\). The response of the system to a shifted input \(x[n-k]\) should be the same as if the output \(y[n]\) has been shifted, i.e., \(y[n-k]\):

\[
\begin{align*}
\text{x[n]} & \quad \xrightarrow{S^k} \quad \text{x[n-k]} \quad \xrightarrow{H} \quad y_1[n] \\
\text{x[n]} & \quad \xrightarrow{H} \quad y[n] \quad \xrightarrow{S^k} \quad y[n-k]
\end{align*}
\]

Response of system to shifted input \(x[n-k]\):
\(y_1[n] = a_0 x[n-k] + a_1 x[n-k+1] + a_2 x[n-k-2] + a_3 x[n-k-3] + a_4 x[n-k-4]\)

Output \(y[n]\) shifted by \(k\):
\(y[n-k] = a_0 x[n-k] + a_1 x[n-k+1] + a_2 x[n-k-2] + a_3 x[n-k-3] + a_4 x[n-k-4]\)

We see that \(y[n-k] = y_1[n]\). Thus, the system is time-invariant.
PROBLEM 9
Consider a series interconnection of system as shown below. The input-output relationship of each system is given by the following equations:

\[ x[n] \xrightarrow{\text{SYSTEM 1}} y_1[n] \xrightarrow{\text{SYSTEM 2}} y_2[n] \xrightarrow{\text{SYSTEM 3}} y[n] \]

System 1: \( y_1[n] = x[-n] \)
System 2: \( y_2[n] = ax[n-1] + b x[n] + cx[n+1] \)
System 3: \( y[n] = x[-n] \)

Here a, b, c are real numbers.

a. Find the input-output relationship for the overall interconnected system.
b. Under what condition (if any) of the values a, b, c, is the overall system linear and time-invariant?
c. Under what condition (if any) of the values a, b, c, is the overall system causal?

**Given the system:**

\[ x[n] \xrightarrow{\text{SYSTEM 1}} w[n] \xrightarrow{\text{SYSTEM 2}} z[n] \xrightarrow{\text{SYSTEM 3}} y[n] \]

\[ y[n] = z[-n] \]
\[ z[n] = aw[n-1] + n^b w[n] + cw[n+1] \]
\[ w[n] = x[-n] \rightarrow w[n-1] = x[-n-1], w[n+1] = x[-n-1] \text{ (the shift is on } n) \]
\[ \rightarrow z[n] = ax[-n+1] + n^b x[-n] + cx[-n-1] \]
\[ \therefore y[n] = z[-n] = ax[n+1] + (-n)^b x[n] + cx[n-1] \]

**b. Linearity:**
If the input to the system is \( kx_a[n] + rx_b[n] \), where \( k, r \), are real numbers, then the output to the system should be \( k y_a[n] + r y_b[n] \), where \( y_a[n], y_b[n] \) are the responses of the system to \( x_a[n] \) and \( x_b[n] \) respectively.

If the input to the system is \( kx_a[n] + rx_b[n] \), then:
\[ y[n] = a(kx_a[n+1] + rx_b[n+1]) + n^b(kx_a[n] + rx_b[n]) + c(kx_a[n-1] + rx_b[n-1]) \]
\[ y[n] = k(ax_a[n+1] + n^b x_a[n] + cx_a[n-1]) + r(ax_b[n+1] + (-n)^b x_b[n] + cx_b[n-1]) \]
\[ \rightarrow y[n] = k(y_a[n]) + r(y_b[n]). \text{ Thus, the system is linear for all real } a, b, c. \]

**Time invariance:**
We have a system \( y[n] = H(x[n]) \). The response of the system to a shifted input \( x[n-k] \) should be the same as if the output \( y[n] \) has been shifted, i.e., \( y[n-k] \):

\[ x[n] \xrightarrow{S^k} x[n-k] \xrightarrow{H} y_1[n] \]
\[ x[n] \xrightarrow{H} y[n] \xrightarrow{S^k} y[n-k] \]

These two should be the same.

Response of system to shifted input \( x[n-k] \):
\[ y_1[n] = ax[n-k+1] + (-n)^b x[n-k] + cx[n-k-1] \]

Output \( y[n] \) shifted by \( k \):
\[ y[n-k] = ax[n-k+1] + (-n+k)^b x[n-k] + cx[n-k-1] \]

For linearity, we need: \( y[n-k] = y_1[n] \). Thus, \((-n)^b = (-n+k)^b \). This only happens when \( b=0 \).

c. Causality requires the term \( x[n+1] \) to go. Thus, we need \( a = 0 \).