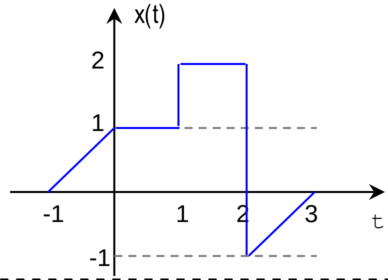
Solutions - Homework # 1

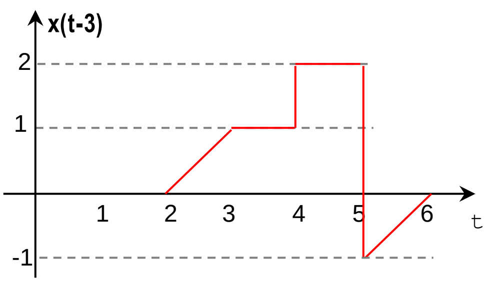
PROBLEM 1

A continuous time signal is shown in the figure. Carefully sketch each of the following signals:

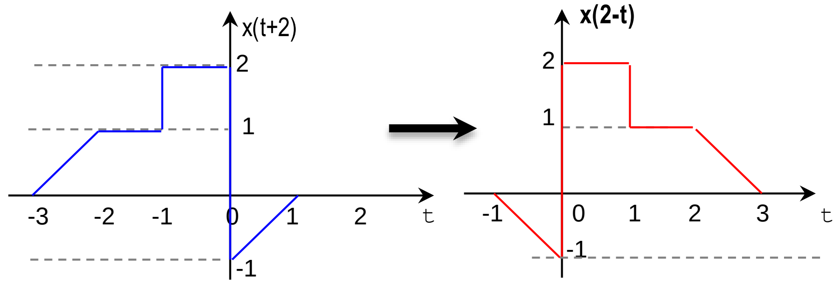
a) x(t-3)	e) x(t)*(δ(t+3/2) - δ(t-3/2))
b) x(2-t)	f) (x(t) + x(2-t))*u(1-t)
c) x(2t+2)	g) x(t/2 - 3) + x(t/3 -2)
d) x(2 - t/3)	h) x(t-1)*u(t-1)



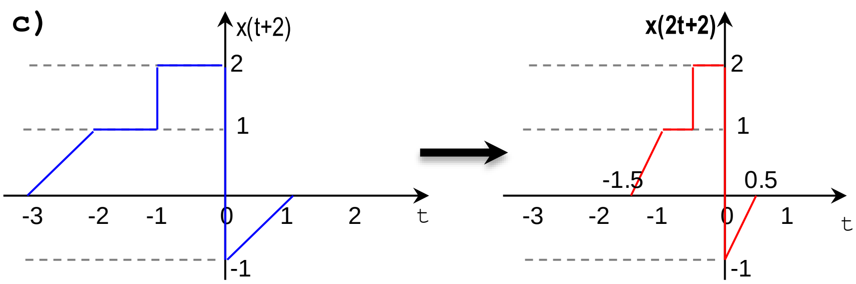
a)



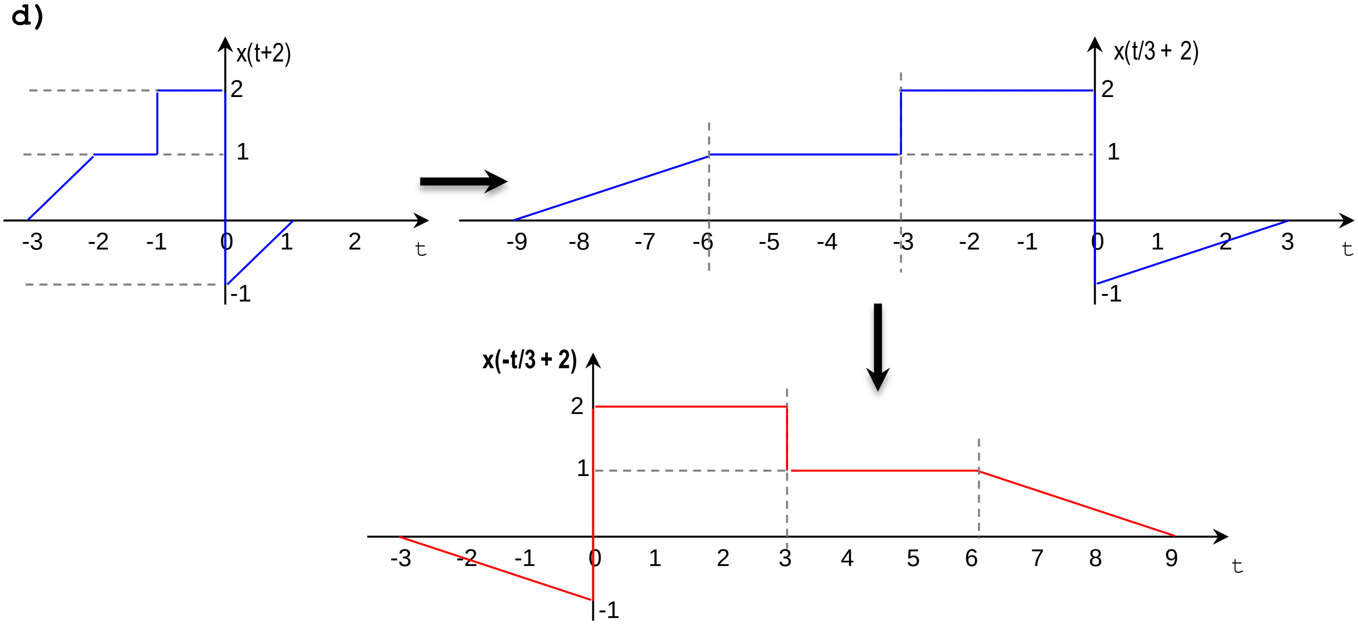
b)



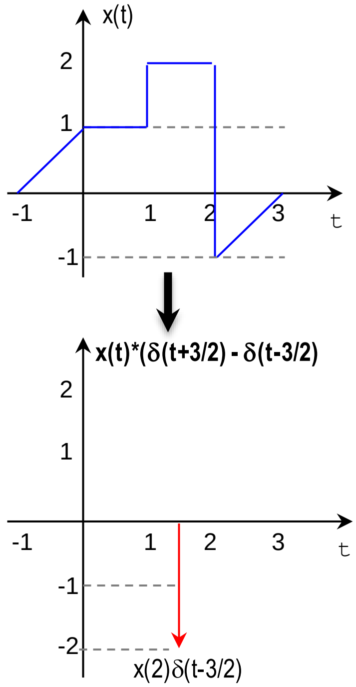
c)



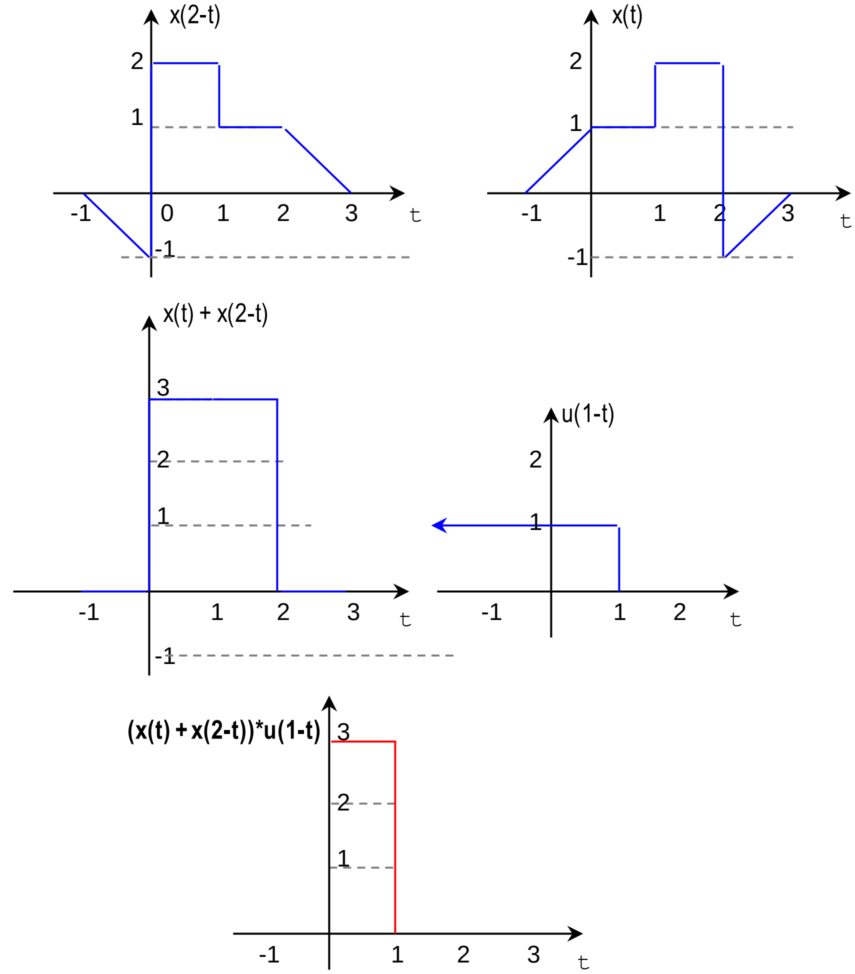
d)



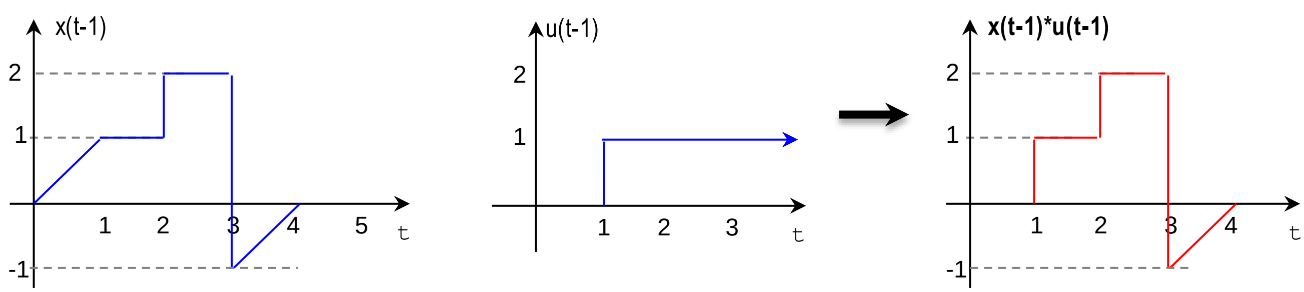
e)


x(t)
x(t)*(δ(t+3/2) - δ(t-3/2)
x(2)δ(t-3/2)

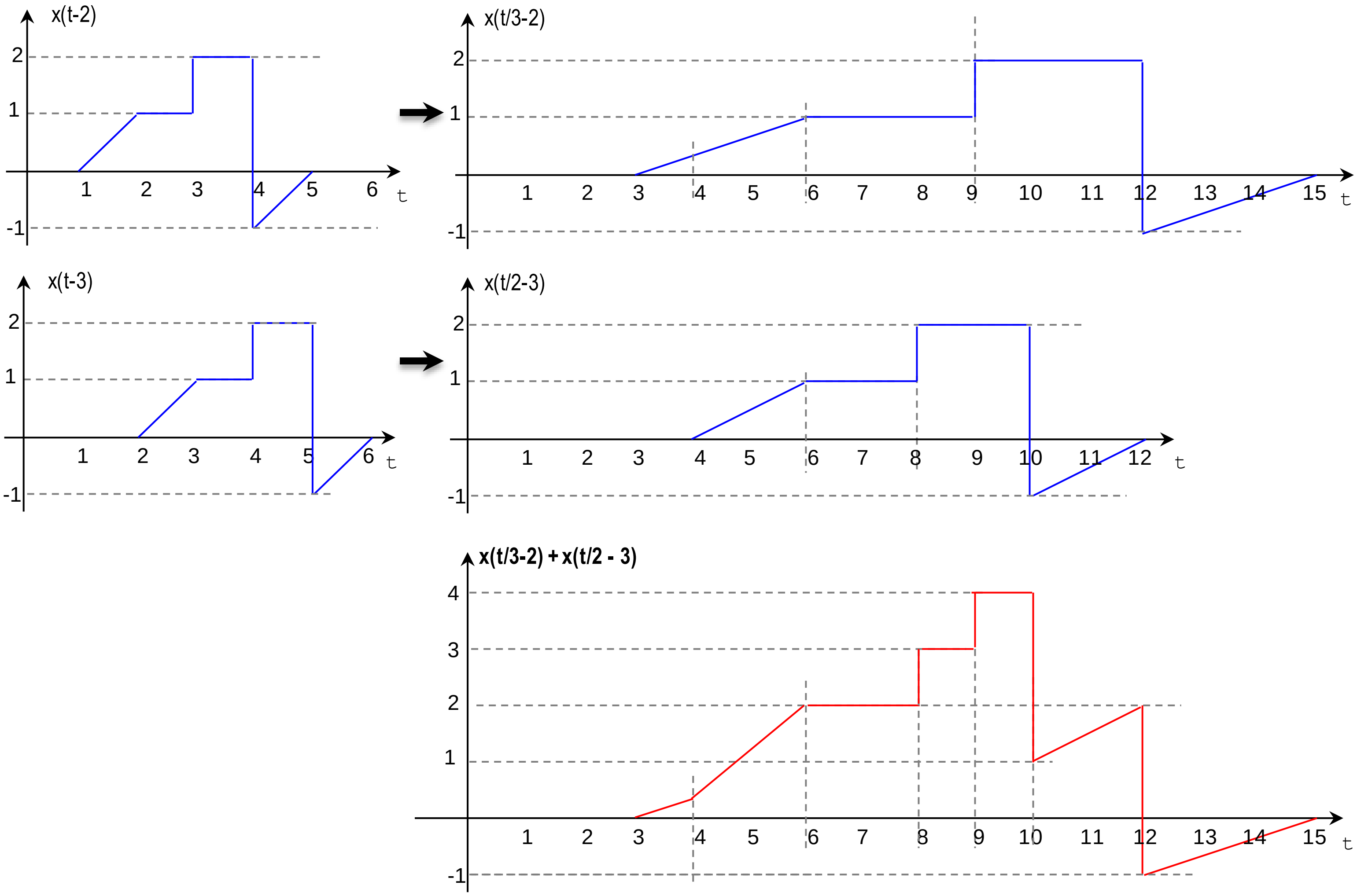
f)


x(2-t)
x(t)
x(t) + x(2-t)
u(1-t)
(x(t) + x(2-t))*u(1-t)

h)

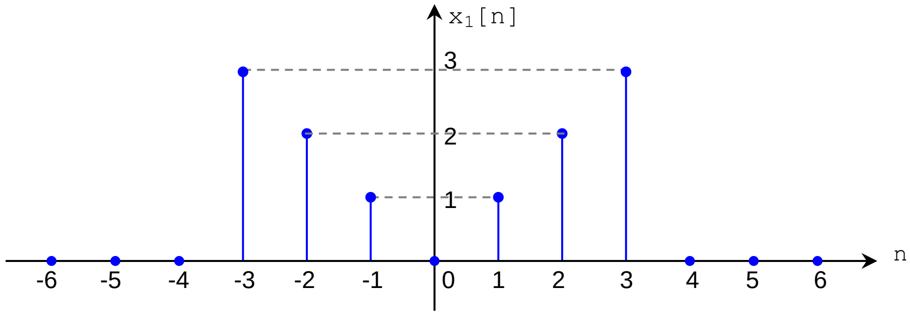

x(t-1)
u(t-1)
x(t-1)*u(t-1)

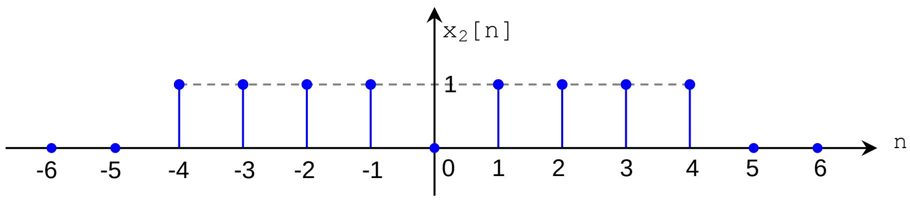
g)



PROBLEM 2

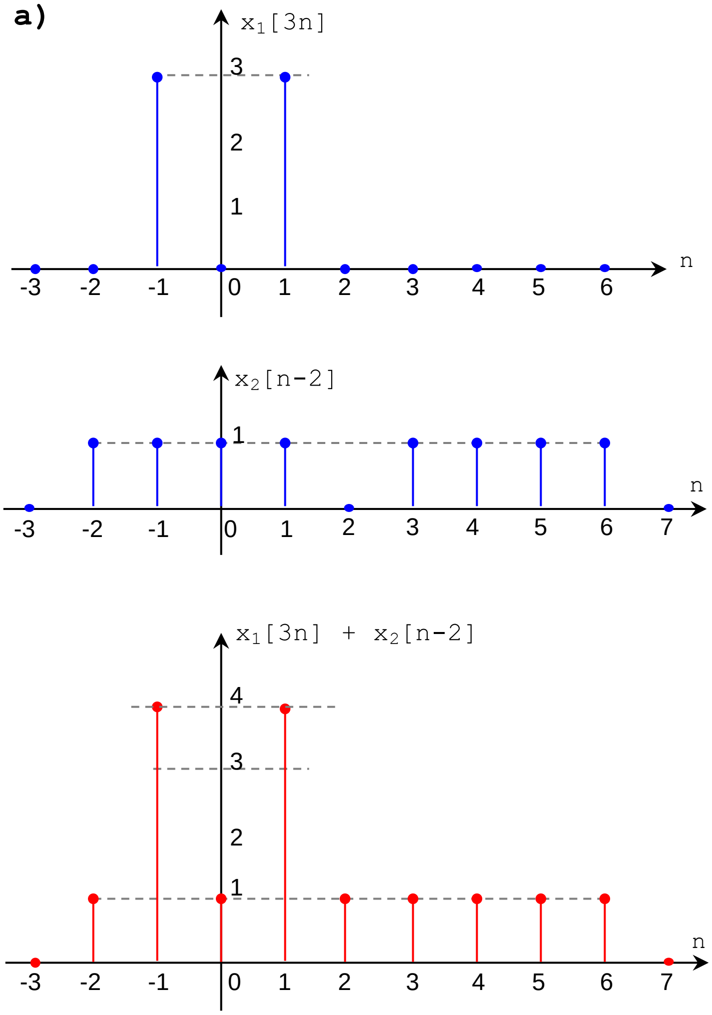
The discrete-time signals $x_1[n]$ and $x_2[n]$ are shown in the figure. Carefully sketch each of the following signals:



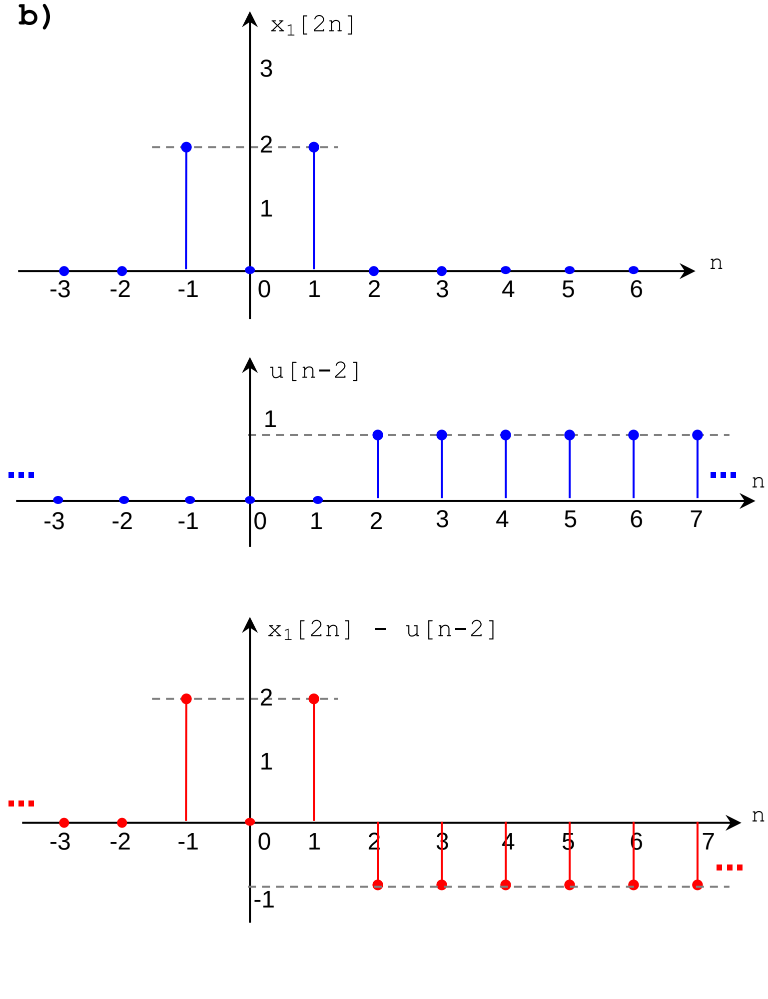


a) $x_1[3n] + x_2[n-2]$	e) $x_1[2n-3]$
b) $x_1[2n] - u[n-2]$	f) $x_1[2n-1] + u[2n+3]$
c) $0.5*x_1[n] + (-1)^n x_1[n]$	g) $x_1[n-1]\delta[n-3]$
d) $x_1[n]*u[1-n]$	h) $2^n x_1[n-1]$

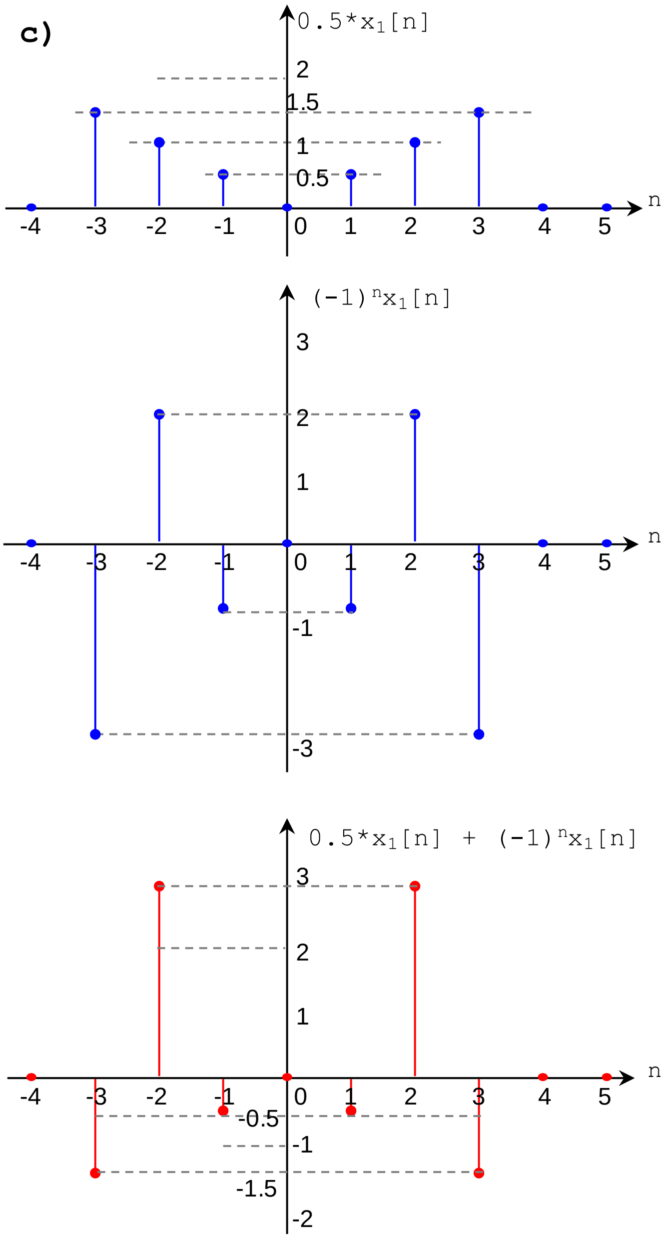
a)



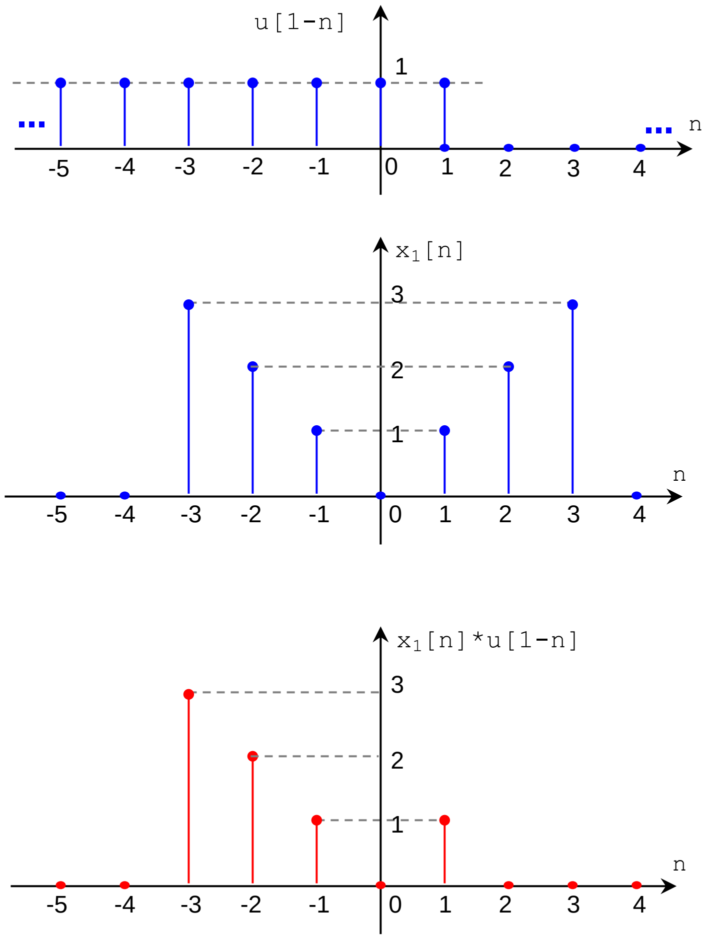
b)



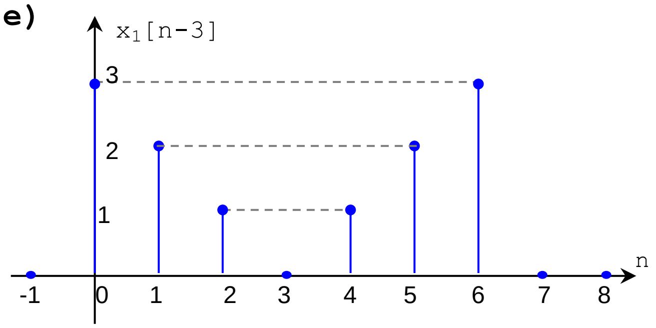
c)

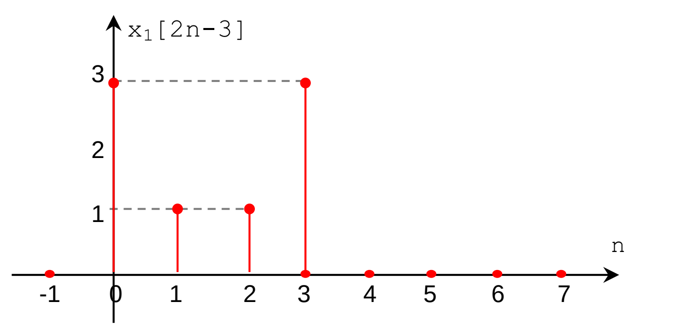


d)

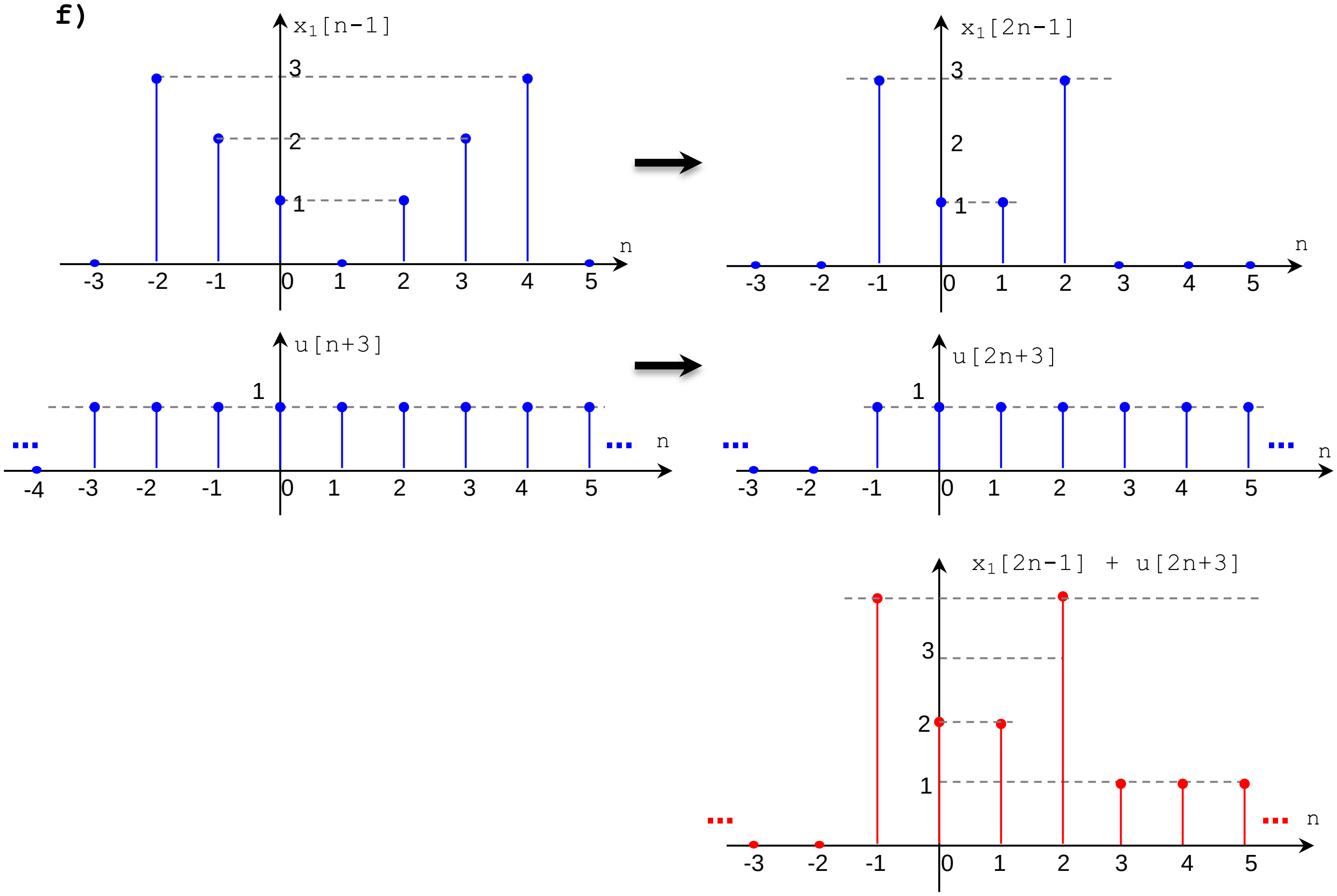


e)

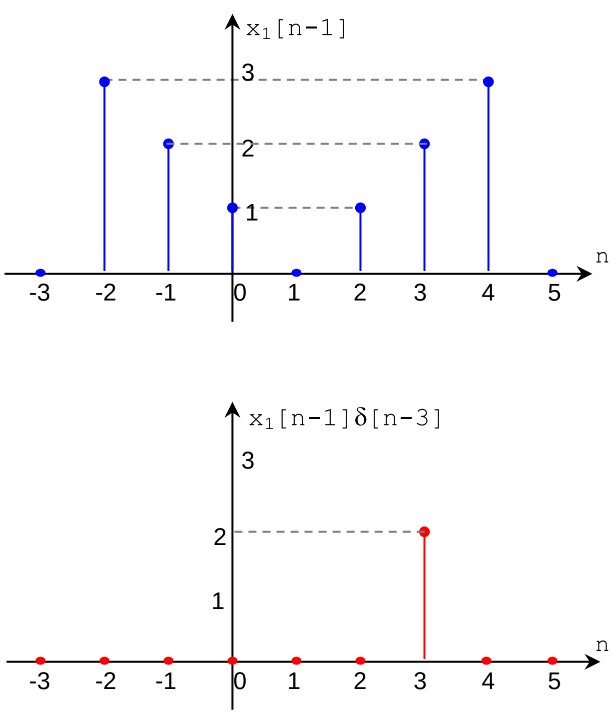




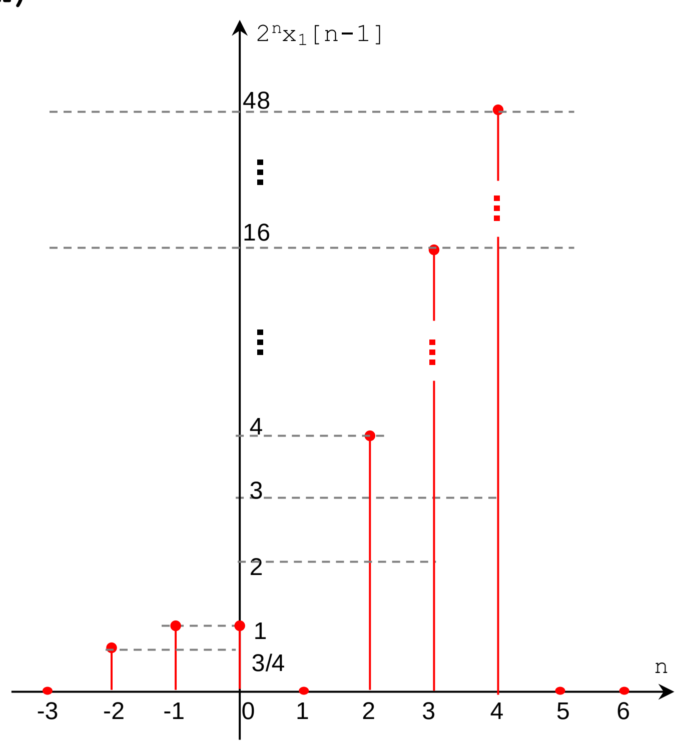
f)


x1[n-1]
x1[2n-1]
u[n+3]
u[2n+3]
x1[2n-1] + u[2n+3]
n

g)


x1[n-1]
x1[n-1]δ[n-3]
n

h)


2^n x1[n-1]
48
16
4
3
2
1
3/4
n

PROBLEM 3

Determine whether the following signals are periodic, and for those which are, find the fundamental period (T for continuous time signals and N for discrete-time signals) and the fundamental angular frequencies (ω for continuous time signals and Ω for discrete-time signals). You must specify the units of these quantities.

a) $x[n] = \cos((8/15)*\pi n)$
b) $x[n] = \sin((7/15)*\pi n)$
c) $x(t) = \sin(2t) + \cos(3t)$
d) $x[n] = \sin((1/5)*\pi n)*\sin((1/3)*\pi n)$
e) $x(t) = \sin(t)u(t-1)$
f) $x(t) = \sin(t)u(t) + \sin(-t)*u(-t)$

---

a) $N = 2\pi m/(8/15\pi) \rightarrow N = (15/4)m$
Then, we choose $m = 4 \rightarrow N = 15$ samples, $\Omega = 2\pi/15$ rads/sample.

b) $N = 2\pi m/(7/15\pi) \rightarrow N = (30/7)m$
Then, we choose $m = 7 \rightarrow N = 30$ samples, $\Omega = \pi/15$ rads/sample.

c) For periodicity: $x(t) = x(t + T)$
$\sin(2t) + \cos(3t) = \sin(2t + 2T) + \cos(3t + 3T)$

We need: $2T = 2\pi k$, and $3T = 2\pi r$, where k, r are integers
Then, we see that: $T = \pi k = (2\pi/3)r \rightarrow 3k = 2r$
The smallest numbers k, r that satisfy this condition are $k = 2$, $r = 3$.

Then, the signal $x(t)$ is periodic with period $T = 2\pi$ secs, $\omega = 1$ rad per second

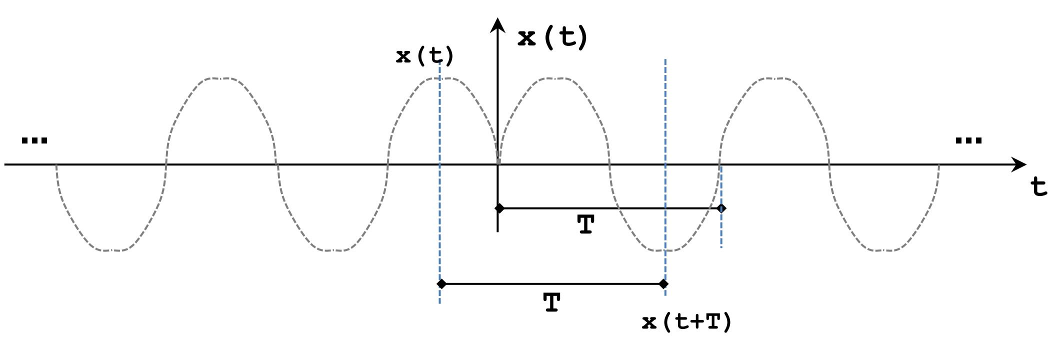
d) For periodicity: $x[n] = x[n + N]$
$\sin(\pi n/5)\sin(\pi n/3) = \sin(\pi n/5 + \pi N/5)\sin(\pi n/3 + \pi N/3)$

We need: $\pi N/5 = 2\pi k$, and $\pi N/3 = 2\pi r$, where k, r are integers
Then, we see that: $N = 10k = 6r \rightarrow 5k = 3r$
The smallest numbers k, r that satisfy this condition are $k = 3$, $r = 5$.

Then, the signal $x[n]$ is periodic with period $N = 30$ samples, $\Omega = \pi/15$ rads/sample.

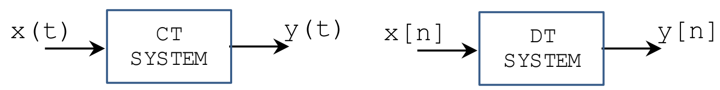
*** Optional:**
If we notice that: $\sin(\pi n/5)\sin(\pi n/3) = \sin(\pi n/5 + \pi k)\sin(\pi n/3 + \pi r)$
Then, we need: $\pi N/5 = \pi k$, and $\pi N/3 = \pi r$, where k, r are integers
Then, $k=3$, $r=5$, and $N = 5k = 3r = 15$ samples, $\Omega = 2\pi/15$ rads/sample.

e) The signal is zero only for $t \leq 1$. Therefore, $x(t)$ is non-periodic.
f) The signal is non-periodic: $x(t) \neq x(t + T)$



PROBLEM 4

The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$ respectively. For each system, determine (and justify) whether it is (i) memoryless, (ii) stable, (iii) causal, (iv) linear, and (v) time invariant. Recall that to disprove that a system has a certain property, all you need is to come up with a counter-example.



a) $y(t) = \cos(x(t))$
b) $y(t) = x(t/2)$
c) $y[n] = 3*x[n]u[n]$
d) $y[n] = \log_2(|x[n]|)$
e) $y[n] = x[n] + x[n-1] + x[n+2]$
f) $y[n] = 2^n x[n]$

---

a) $y(t) = \cos(x(t))$

It is <u>memoryless</u>: it only depends on the current value of the signal.

<u>Stability</u>:
If $|x(t)| \leq Mx < \infty, \ \forall t \rightarrow$ it should be that $|y(t)| \leq My < \infty, \ \forall t$
$|y(t)| = |\cos(x(t)| \rightarrow |y(t)| = |\cos(x(t)| \leq 1$

Therefore, the system is stable.

<u>Causality</u>: It is causal, because it does not depend on future values of the input $x(t)$.

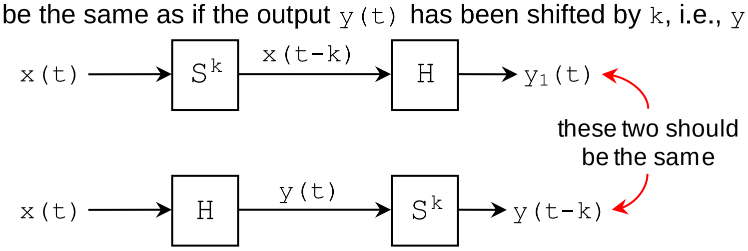
<u>Linearity</u>:
If the input to the system is $ax_A(t) + bx_B(t)$, where a,b, are real numbers, then the output to the system should be $ay_A(t) + by_B(t)$, where $y_A(t)$, $y_B(t)$ are the responses of the system to $x_A(t)$ and $x_B(t)$ respectively.

If the input to the system is $ax_A(t) + bx_B(t)$, then:
$y(t) = \cos(ax_A(t) + bx_B(t))$

$ay_A(t) + by_B(t) = a\cos(x_A(t)) + b\cos(x_B(t))$

We see that $y(t) \neq a(y_A(t)) + b(y_B(t))$. Thus, the system is NOT linear.

<u>Time invariance</u>:
We have a system $y(t) = H(x(t))$. The response of the system to a shifted input $x(t-k)$ should be the same as if the output $y(t)$ has been shifted by k, i.e., $y(t-k)$:



Response of system to shifted input $x(t-k)$: $y_1(t) = \cos(x(t-k))$

Output $y(t)$ shifted by k: $y(t-k) = \cos(x(t-k))$

We see that $y(t-k) = y_1(t)$. Thus, the system is time invariant.

b) $y(t) = x(t/2)$
It is NOT memoryless: $y(2) = x(1)$, $y(3) = x(1.5)$.

Stability:
If $|x(t)| \le Mx < \infty, \forall t \rightarrow$ it should be that $|y(t)| \le My < \infty, \forall t$
$|y(t)| = |x(t/2)| \le |x(t)|$

Therefore, the system is stable.

Causality: It is NOT causal, because it depends on future values of the input $x(t)$:
For example: $y(-1) = x(-0.5)$, $y(-4) = x(-2)$.

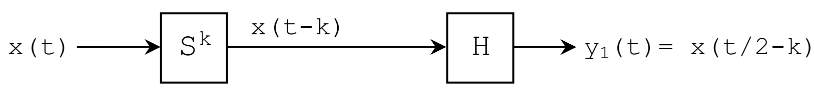
Linearity:
If the input to the system is $ax_A(t) + bx_B(t)$, where a, b, are real numbers, then the output to the system should be $ay_A(t) + by_B(t)$, where $y_A(t)$, $y_B(t)$ are the responses of the system to $x_A(t)$ and $x_B(t)$ respectively.

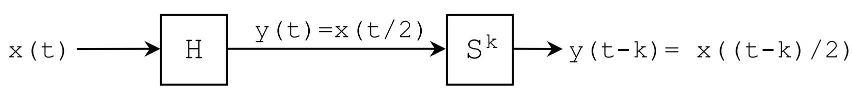
If the input to the system is $ax_A(t) + bx_B(t)$, then:
$y(t) = ax_A(t/2) + bx_B(t/2))$

$ay_A(t) + by_B(t) = ax_A(t/2) + bx_B(t/2)$

We see that $y(t) = ay_A(t) + by_B(t)$. Thus, the system is linear.

Time invariance:
We have a system $y(t) = H(x(t)) = x(t/2)$. The response of the system to a shifted input $x(t-k)$ should be the same as if the output $y(t)$ has been shifted by k, i.e., $y(t-k)$.





Response of system to shifted input $x(t-k)$: $y_1(t) = x(t/2-k)$

Output $y(t)$ shifted by k: $y(t-k) = x((t-k)/2)$

We see that $y(t-k) \neq y_1(t)$. Thus, the system is NOT time invariant.

c) $y[n] = 3*x[n]u[n]$
It is memoryless: it only depends on the current sample of the signal.

Stability:
If $|x[n]| \le Mx < \infty, \forall n \rightarrow$ it should be that $|y[n]| \le My < \infty, \forall n$
$|y[n]| = |3x[n]u[n]| \le |3x[n]|$

Therefore, the system is stable.

Causality: It is causal, because it does not depend on future samples of the input $x[n]$.

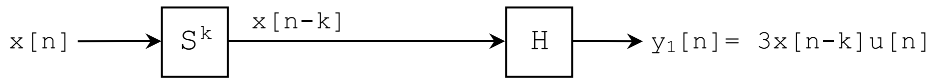
Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a, b, are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

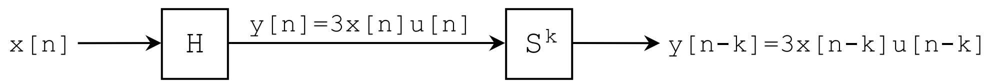
If the input to the system is $ax_A[n] + bx_B[n]$, then:
$y[n] = 3(ax_A[n] + bx_B[n])u[n]$
$y[n] = 3(ax_A[n]u[n]) + 3(bx_B[n])u[n])$

$ay_A[n] + by_B[n] = a*3*x_A[n]u[n] + b*3*x_B[n]$

We see that $y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

Time invariance:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by k, i.e., $y[n-k]$:





Response of system to a shifted input $x[n-k]$: $y_1[n] = 3x[n-k]u[n]$

Output $y[n]$ shifted by k: $y[n-k] = 3x[n-k]u[n-k]$

We see that $y[n-k] \neq y_1[n]$. Thus, the system is NOT time invariant.

d) $y[n] = \log_2(|x[n]|)$
It is memoryless: it only depends on the current sample of the signal.

Stability:
If $|x[n]| \leq Mx < \infty, \forall n \rightarrow$ it should be that $|y[n]| \leq My < \infty, \forall n$
$|y[n]| = |\log_2|x[n]||$

When $x[n]$ approaches 0, $\log2(|x[n]|)$ approaches towards minus infinity. Thus, the system is NOT stable.

Causality: It is causal, because it does not depend on future samples of the input $x[n]$.

Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a,b, are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $ax_A[n] + bx_B[n]$, then:
$y[n] = \log_2(|ax_A[n] + bx_B[n]|)$

$ay_A[n] + by_B[n] = a\log2(|x_A[n]|) + b\log_2(|x_B[n]|)$

We see that $y[n] \neq a(y_A[n]) + b(y_B[n])$. Thus, the system is NOT linear.

Time invariance:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by k, i.e., $y[n-k]$:

Response of system to a shifted input $x[n-k]$: $y_1[n] = \log_2(|x[n-k]|)$

Output $y[n]$ shifted by k: $y[n-k] = \log_2(|x[n-k]|)$

We see that $y[n-k] = y_1[n]$. Thus, the system is time invariant.

e) $y[n] = x[n] + x[n-1] + x[n+2]$
It is NOT memoryless: it depends on previous and future samples of the signal.

Stability:
If $|x[n]| \leq Mx < \infty, \forall n \rightarrow$ it should be that $|y[n]| \leq My < \infty, \forall n$
$|y[n]| = |x[n] + x[n-1] + x[n+2]| \leq |x[n]| + |x[n-1]| + |x[n+2]|$

If $|x[n]| \leq Mx < \infty, \forall n \rightarrow |x[n-k]| \leq Mx < \infty, \forall n$
Then:
$|y[n]| \leq Mx + Mx + Mx$
$|y[n]| \leq 3Mx$

Therefore, the system is stable.

Causality: It is NOT causal, because it depends on future samples of the input $x[n]$.

Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a, b, are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $ax_A[n] + bx_B[n]$, then:
$y[n] = (ax_A[n] + bx_B[n]) + (ax_A[n-1] + bx_B[n-1]) + (ax_A[n+2] + bx_B[n+2])$

$y[n] = a(x_A[n] + x_A[n-1] + x_A[n+2]) + b(x_B[n] + x_B[n-1] + x_B[n+2])$

$\rightarrow y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

Time invariance:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by k, i.e., $y[n-k]$:

Response of system to a shifted input $x[n-k]$: $y_1[n] = x[n-k] + x[n-k-1] + x[n-k+2]$

Output $y[n]$ shifted by k: $y[n-k] = x[n-k] + x[n-k-1] + x[n-k+2]$

We see that $y[n-k] = y_1[n]$. Thus, the system is time invariant.

f) $y[n] = 2^n x[n]$
It is memoryless: it only depends on the current sample of the signal.

Stability:
If $|x[n]| \leq Mx < \infty, \forall n \rightarrow$ it should be that $|y[n]| \leq My < \infty, \forall n$
$|y[n]| = |2^n x[n]| \leq 2^n |x[n]|$

As n grows, 2^n tends to infinity. Therefore, the system is NOT stable.

Causality: It is causal, because it does not depend on future samples of the input $x[n]$.

Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a, b, are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

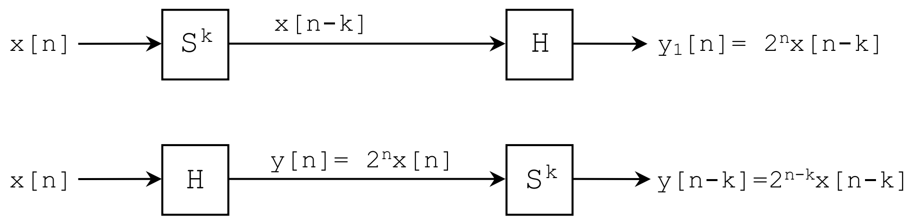
If the input to the system is $ax_A[n] + bx_B[n]$, then:

$y[n] = 2^n(ax_A[n] + bx_B[n])$
$y[n] = 2^n(ax_{A[}n]) + 2^n(bx_B[n])$

$ay_A[n] + by_B[n] = a*2^n x_A[n]u[n] + b*2^n x_B[n]$

We see that $y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

<u>Time invariance</u>:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted by k, i.e., $y[n-k]$:



Response of system to a shifted input $x[n-k]$: $y_1[n] = 2^n x[n-k]$

Output $y[n]$ shifted by k: $y[n-k] = 2^{n-k} x[n-k]$

We see that $y[n-k] \neq y_1[n]$. Thus, the system is NOT time invariant.

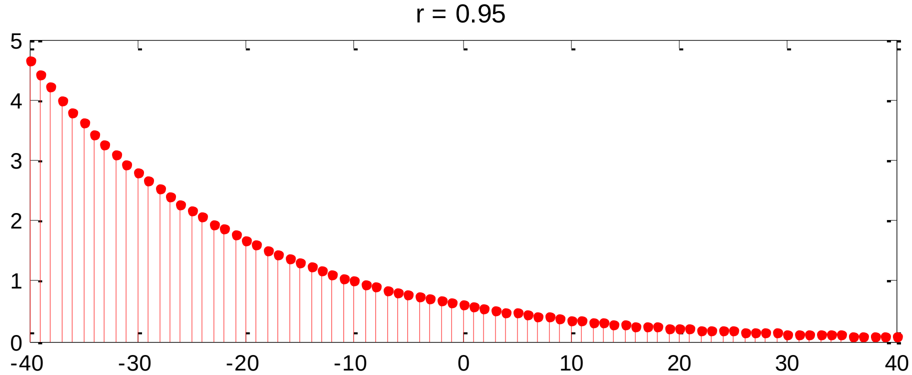
PROBLEM 5

Using MATLAB®, plot (with the command 'stem') the following signals for $n = -40$ to 40. Attach your MATLAB code to the plots.

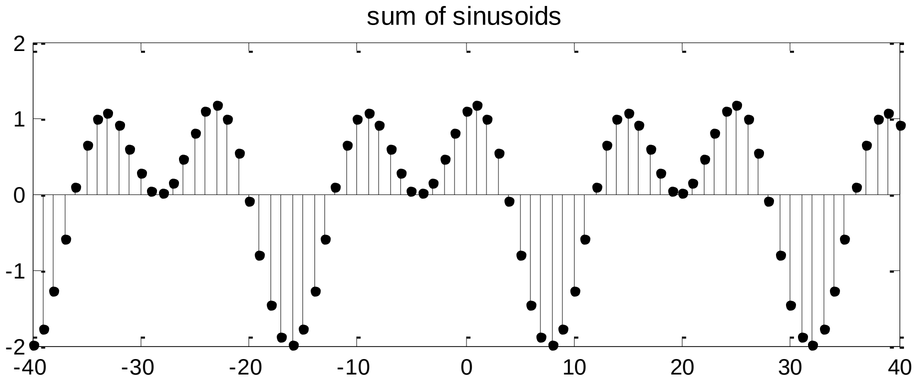
a) $x[n] = 0.6*(0.95)^n$
b) $x[n] = \cos((\pi/12)*n + \pi/3) + \sin((\pi/6)*n + \pi/5)$
c) $x[n] = A*\cos(\Omega_0 n + \phi)$ for:
   i. $A = 2.5,\ \Omega_0 = 2\pi/45,\ \phi = \pi/5$
   ii. $A = 0.5,\ \Omega_0 = \pi/12,\ \phi = \pi/3$
   iii. $A = 1.5,\ \Omega_0 = \pi/2,\ \phi = \pi/5$

```
clear all; close all ; clc;
% Generating vector of samples:
n = -40:40; % [-40 -19.... 40]

% a) x[n] = 0.6*(0.95)^n
B = 0.6; r = 0.95;
x1 = B*(r.^n);
figure; stem (n,x1,'.r'); title ('r = 0.95'); xlabel ('n');

% b) x[n] = cos(pi*n/12 + pi/3) + sin(pi*n/6 + pi/5)
x2 = cos(pi*n/12 + pi/3) + sin(pi*n/6 + pi/5);
figure; stem (n,x2,'.k'); title ('sum of sinusoids'); xlabel ('n');

% c) x[n] = Acos(omega0*n + phi):
  A = 2.5; omega0 = 2*pi/45; phi = pi/5;
    xa = A*cos(omega0*n + phi);
  A = 0.5; omega0 = pi/12; phi = pi/3;
    xb = A*cos(omega0*n + phi);
  A = 1.5; omega0 = pi/2; phi = pi/5;
    xc = A*cos(omega0*n + phi);
figure; stem (n,xa,'.b'); title ('Sinusoid. A=2.5, omega0=2\pi/45, phi= \pi/5');
xlabel ('n');
figure; stem (n,xb,'.b'); title ('Sinusoid. A=0.5, omega0=\pi/2, phi= \pi/3');
xlabel ('n');
figure; stem (n,xc,'.b'); title ('Sinusoid. A=1.5, omega0 = \pi/12, phi = \pi/5');
xlabel ('n');
```

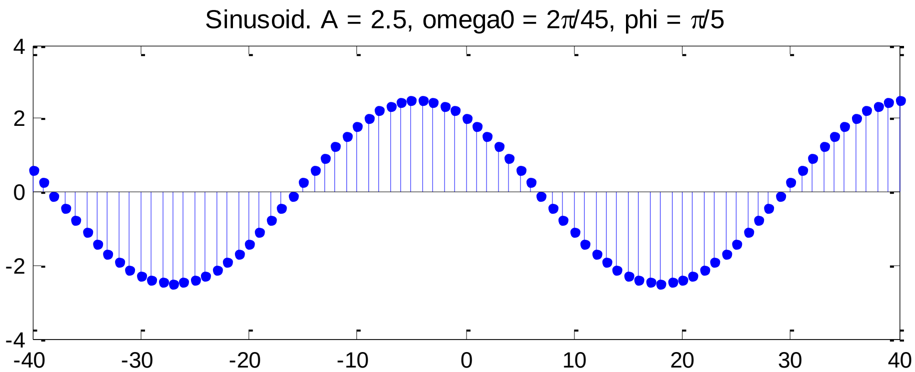
a)



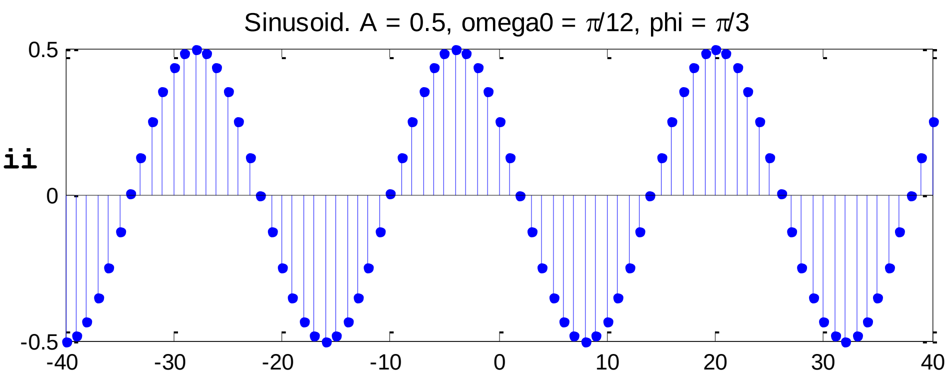
b)



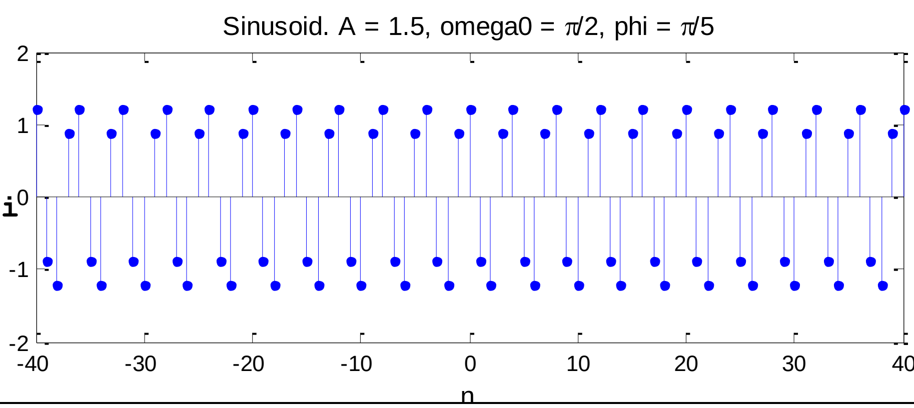
c) i



c) ii



c) iii



PROBLEM 6

Let x(t) be the continuous-time complex exponential signal:

$$x(t) = \exp(j\omega_0 t)$$

with fundamental frequency $\omega = \omega_0$, and fundamental period $T_0 = 2\pi/\omega_0$.

The discrete-time signal $x[n]$ was generated by uniformly sampling (taking equally spaced samples) the signal $x(t)$ with a sampling period T_S (in seconds)

$$x[n] = x(nTs) = \exp(j\omega_0 nT_S)$$

a. Show that $x[n]$ is periodic if and only if T_S/T_0 is a rational number.
b. If $\omega_0 = \pi/8$, $N = 40$, what is the minimum number of cycles of the original complex exponential (also called envelope cycles) that are required for $x[n]$ to be periodic?
c. Once you obtained the minimum number of envelope cycles, what is the sampling period (in seconds)?

---

a. $\exp(j\omega_0 nT_S) = \cos(\omega_0 nT_S) + j\sin(\omega_0 nT_S)$
The complex exponential has the same periodicity as the cosine and sine functions:
$x[n] = x[n + N] \rightarrow \exp(j\omega_0 nT_S) = \exp(j\omega_0 (n+N)T_S) = \exp(j\omega_0 nT_S)\exp(j\omega_0 NT_S)$
$\rightarrow \omega_0 NT_S = 2\pi m,\ \omega_0 = 2\pi/T_0 \rightarrow NT_S/T_0 = m$

For $NT_S/T_0 = m$ to hold, i.e., for m to be an integer, T_S/T_0 has to be a rational number.

b. $N = 40,\ \omega_0 = \pi/8 \rightarrow T_0 = 16$ secs.
$NT_S/T_0 = m \rightarrow 40*(T_S/16) = m \rightarrow T_S/16 = m/40$
Any integer m will make $T_S/16$ a rational number, so we pick $m = 1$.

c. $Ts/16 = 1/40 \rightarrow Ts = 16/40 = 0.4$ secs.

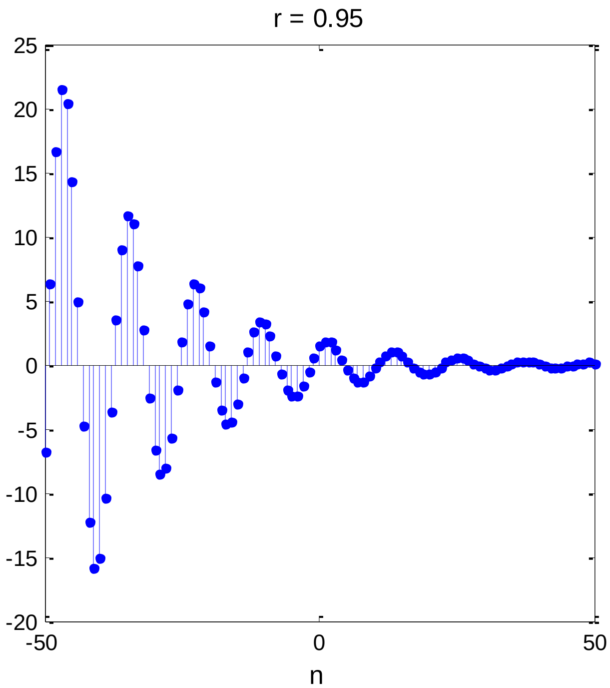
PROBLEM 7

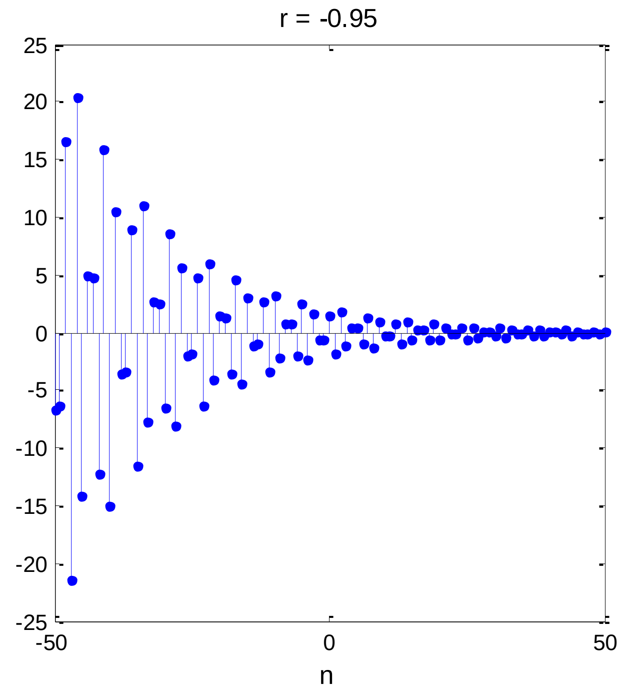
Using MATLAB®, plot (with the 'stem' command) the following exponentially damped sinusoidal signal for two different values of r (one positive and one negative).

$$x[n] = Br^n \sin(\Omega_0 n + \phi)$$

Note that $0 < |r| < 1$ (otherwise there is no exponential decay).
Range: $n = -50$ to 50, Fixed parameters: $\phi = \pi/4,\ B = 2$.
Pick Ω_0 and r judiciously so that a clear damping on the sinusoid can be seen in the plot. Attach your MATLAB code to the plots.

```
clear all; close all; clc;

n = -50:50;
B = 2;
phi = pi/4;

omega0 = pi/6;
r = 0.95; % 0 < r < 1: decaying exponential

x = B*(r.^n).*sin(omega0*n + phi);
figure; stem (n,x,'.b'); title ('r = 0.95'); xlabel ('n');

z = B*((-r).^n).*sin(omega0*n + phi);
figure; stem (n,z,'.b'); title ('r = 0.95'); xlabel ('n');
```





PROBLEM 8

The output of a discrete-time system is related to its input $x[n]$ as follows:

$$y[n] = a_0 x[n] + a_1 x[n+1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]$$

where $a_0,\ a_1,\ a_2,\ a_3,\ a_4$ are real values.

Let the operator S^k denote a system that shifts the input $x[n]$ by k samples to produce $x[n-k]$.

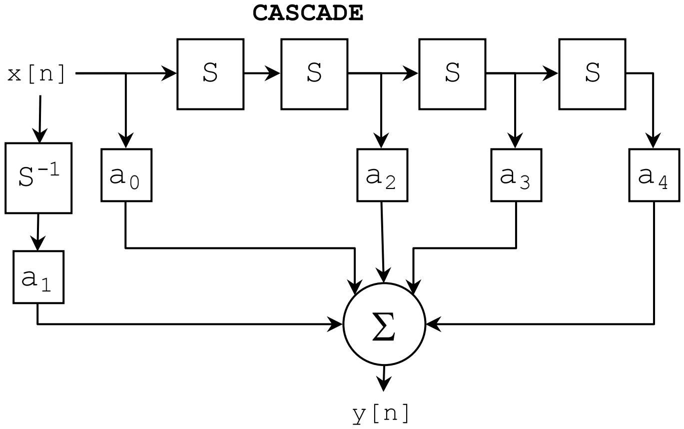
a. Formulate the operator H for the system relating $y[n]$ to $x[n]$. Then develop a block diagram representation for H, using (i) cascade implementation, and (ii) parallel implementation.
b. Demonstrate that the system is BIBO stable for all $a_0,\ a_1,\ a_2,\ a_3,\ a_4$ (real values)
c. Under what condition (if any) of the values $a_0,\ a_1,\ a_2,\ a_3,\ a_4$ is the system causal?
d. Demonstrate that the system is linear and time-invariant.

---

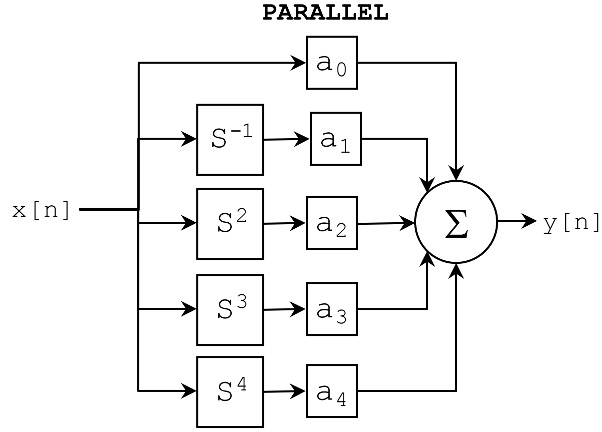
a. A system $y[n] = ax[n-k]$ is represented by $H = aS^k$.
Then, the system $y[n] = a_0 x[n] + a_1 x[n+1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]$ is represented by $H = a_0S^0 + a_1S^{-1} + a_2S^2 + a_3S^3 + a_4S^4$.

$\rightarrow H = a_0 + a_1S^{-1} + a_2S^2 + a_3S^3 + a_4S^4$.

CASCADE



PARALLEL



b. Stability:
If $|x[n]| \le Mx < \infty, \forall n \rightarrow$ it should be that $|y[n]| \le My < \infty, \forall n$

$$|y[n]| = |a_0 x[n] + a_1 x[n+1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]|$$
$$|y[n]| \le |a_0 x[n]| + |a_1 x[n+1]| + |a_2 x[n-2]| + |a_3 x[n-3]| + |a_4 x[n-4]|$$

$$|y[n]| \le |a_0||x[n]| + |a_1||x[n+1]| + |a_2||x[n-2]| + |a_3||x[n-3]| + |a_4||x[n-4]|$$

If $|x[n]| \le Mx < \infty, \forall n \rightarrow |x[n-k]| \le Mx < \infty, \forall n$
Then:
$$|y[n]| \le |a_0|Mx + |a_1|Mx + |a_2|Mx + |a_3|Mx + |a_4|Mx$$
$$|y[n]| \le (|a_0| + |a_1| + |a_2| + |a_3| + |a_4|)Mx$$

Therefore, $y[n]$ is stable.

c. The term $x[n+1]$ makes the system noncausal. Then, for the system to be causal, we require that $a_1 = 0$.

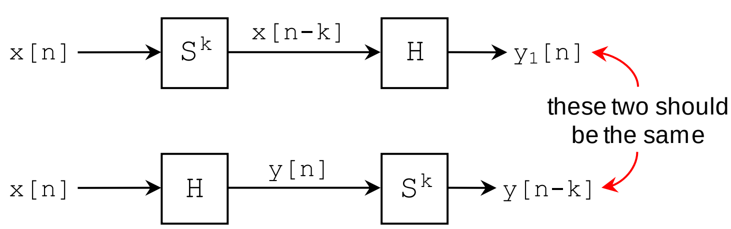
d. Linearity:
If the input to the system is $ax_A[n] + bx_B[n]$, where a, b, are real numbers, then the output to the system should be $ay_A[n] + by_B[n]$, where $y_A[n]$, $y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $ax_A[n] + bx_B[n]$, then:
$$y[n] = a_0(ax_A[n] + bx_B[n]) + a_1(ax_A[n+1] + bx_B[n+1]) + a_2(ax_A[n-2] + bx_B[n-2]) + a_3(ax_A[n-3] + bx_B[n-3]) + a_4(ax_A[n-4] + bx_B[n-4])$$

$$y[n] = a(a_0 x_A[n] + a_1 x_A[n+1] + a_2 x_A[n-2] + a_3 x_A[n-3] + a_4 x_A[n-4]) + b(a_0 x_B[n] + a_1 x_B[n+1] + a_2 x_B[n-2] + a_3 x_B[n-3] + a_4 x_B[n-4])$$

$\rightarrow y[n] = a(y_A[n]) + b(y_B[n])$. Thus, the system is linear.

Time invariance:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output $y[n]$ has been shifted, i.e., $y[n-k]$:



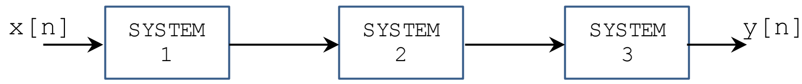
Response of system to shifted input $x[n-k]$:
$$y_1[n] = a_0 x[n-k] + a_1 x[n-k+1] + a_2 x[n-k-2] + a_3 x[n-k-3] + a_4 x[n-k-4]$$

Output $y[n]$ shifted by k:
$$y[n-k] = a_0 x[n-k] + a_1 x[n-k+1] + a_2 x[n-k-2] + a_3 x[n-k-3] + a_4 x[n-k-4]$$

We see that $y[n-k] = y_1[n]$. Thus, the system is time-invariant.

PROBLEM 9

Consider a series interconnection of system as shown below. The input-output relationship of each system is given by the following equations:



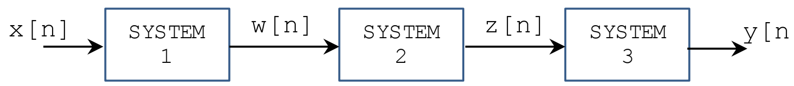
System 1: $y[n] = x[-n]$
System 2: $y[n] = ax[n-1] + n^b x[n] + cx[n+1]$
System 3: $y[n] = x[-n]$

Here a, b, c are real numbers.

a. Find the input-output relationship for the overall interconnected system.
b. Under what condition (if any) of the values a, b, c , is the overall system linear and time-invariant?
c. Under what condition (if any) of the values a, b, c , is the overall system causal?

---

a. Given the system:



$y[n] = z[-n]$
$z[n] = aw[n-1] + n^b w[n] + cw[n+1]$
$w[n] = x[-n] \rightarrow w[n-1] = x[-n+1],\ w[n+1] = x[-n-1]$ (the shift is on n).
$\rightarrow z[n] = ax[-n+1] + n^b x[-n] + cx[-n-1]$
$\therefore y[n] = z[-n] = ax[n+1] + (-n)^b x[n] + cx[n-1]$

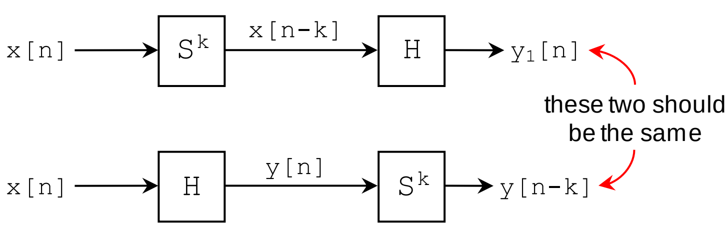
b. <u>Linearity</u>:
If the input to the system is $kx_A[n] + rx_B[n]$, where k, r, are real numbers, then the output to the system should be $ky_A[n] + ry_B[n]$, where $y_A[n],\ y_B[n]$ are the responses of the system to $x_A[n]$ and $x_B[n]$ respectively.

If the input to the system is $kx_A[n] + rx_B[n]$, then:
$y[n] = a(kx_A[n+1] + rx_B[n+1]) + n^b(kx_A[n] + rx_B[n]) +$
$\quad c(kx_A[n-1] + rx_B[n-1])$

$y[n] = k(ax_A[n+1]+ n^b x_A[n]+ cx_A[n-1])+ r(ax_B[n+1] + (-n)^b x_B[n] + cx_B[n-1])$

$\rightarrow y[n] = k(y_A[n]) + r(y_B[n])$. Thus, the system is linear for all real a, b, c.

<u>Time invariance</u>:
We have a system $y[n] = H(x[n])$. The response of the system to a shifted input $x[n-k]$ should be the same as if the output y[n] has been shifted, i.e., $y[n-k]$:



Response of system to shifted input $x[n-k]$:
$y_1[n] = ax[n-k+1] + (-n)^b x[n-k] + cx[n-k-1]$

Output $y[n]$ shifted by k:
$y[n-k] = ax[n-k+1] + (-n+k)^b x[n-k] + cx[n-k-1]$

For linearity, we need: $y[n-k] = y1[n]$. Thus, $(-n)^b = (-n+k)^b$. This only happens when $b=0$.

c. Causality requires the term $x[n+1]$ to go. Thus, we need $a = 0$.