Solutions - Final Exam

PROBLEM 1 (15 PTS)

a) For the following Fourier Transform of a periodic signal: Determine the fundamental angular frequency and the Fourier series coefficients. Then determine the corresponding time-domain signal.

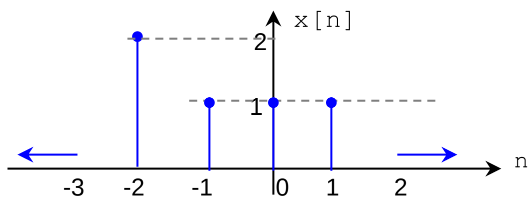
$$X(j\omega) = j\delta\left(\omega - \frac{\pi}{3}\right) + 2\delta\left(\omega - \frac{\pi}{7}\right)$$

b) Determine the Fourier Series, the fundamental angular frequency, and the Fourier Transform of the following signal:

$$x(t) = Cos(\pi t) + 3jSin(2\pi t)$$

c) Determine the Discrete Time Fourier Series and the fundamental angular frequency of the following signal (sketch one period of $X[k]$). Then, determine the DTFT for one period (provide the equation and sketch it).

i. $x[n]$ with one period shown ⇒



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a) $X(j\omega) = j\delta\left(\omega - \frac{\pi}{3}\right) + 2\delta\left(\omega - \frac{\pi}{7}\right)$

$\frac{\pi}{3} = k_1\omega_0, \quad \frac{\pi}{7} = k_2\omega_0 \rightarrow \omega_0 = \frac{\pi}{3k_1} = \frac{\pi}{7k_2} \rightarrow 7k_2 = 3k_1 \rightarrow k_2 = 3, k_1 = 7 \Rightarrow \omega_0 = \frac{\pi}{21}$

$X(j\omega) = 2\pi\left\{X[7]\delta\left(\omega - 7\frac{\pi}{21}\right) + X[3]\delta\left(\omega - 3\frac{\pi}{21}\right)\right\}$

Then:

$$X[k] = \begin{cases} \frac{j}{2\pi}, & k = 7 \\ \frac{1}{\pi}, & k = 3 \\ 0, & otherwise \end{cases}$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{j}{2\pi}e^{j\frac{\pi}{3}t} + \frac{1}{\pi}e^{j\frac{\pi}{7}t}$$

b) $x(t) = Cos(\pi t) + 3jSin(2\pi t)$

$\pi T = 2\pi r, 2\pi T = 2\pi k \rightarrow T = 2r = k \rightarrow r = 1, k = 2 \Rightarrow T = 2, \omega_0 = \pi$

$x(t) = \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} + \frac{3}{2}e^{j2\pi t} - \frac{3}{2}e^{-j2\pi t} = X[1]e^{j\pi t} + X[-1]e^{-j\pi t} + X[2]e^{j2\pi t} + X[-2]e^{-j2\pi t}$

Thus:

$$X[k] = \begin{cases} \frac{1}{2}, k = \pm 1 \\ \frac{3}{2}, k = 2 \\ -\frac{3}{2}, k = -2 \\ 0, otherwise \end{cases}$$

$$\therefore X(j\omega) = \pi\delta(\omega - \pi) + \pi\delta(\omega + \pi) + 3\pi\delta(\omega - 2\pi) - 3\pi\delta(\omega + 2\pi)$$

c) We first determine the DTFS: $N = 4, \Omega_0 = \frac{\pi}{2}$

$X[k] = \frac{1}{N}\sum_{N} x[n]e^{-jk\Omega_0 n} = \frac{1}{4}\sum_{n=-2}^{1} x[n]e^{-jk\Omega_0 n} = \frac{1}{4}\left(2e^{jk\pi} + e^{jk\frac{\pi}{2}} + 1 + e^{-jk\frac{\pi}{2}}\right)$

$X[k] = \frac{1}{4}\left(2e^{jk\pi} + 2Cos\left(k\frac{\pi}{2}\right) + 1\right), \quad -2 \le k \le 1$

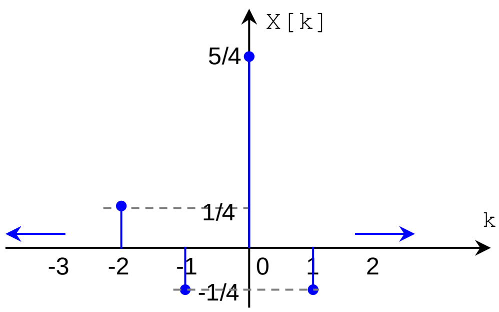
$X[-2] = \frac{1}{4}\left(2e^{-j2\pi} + 2Cos\left(-2\frac{\pi}{2}\right) + 1\right) = \frac{1}{4}, \quad X[-1] = \frac{1}{4}\left(2e^{-j\pi} + 2Cos\left(-\frac{\pi}{2}\right) + 1\right) = -\frac{1}{4}$

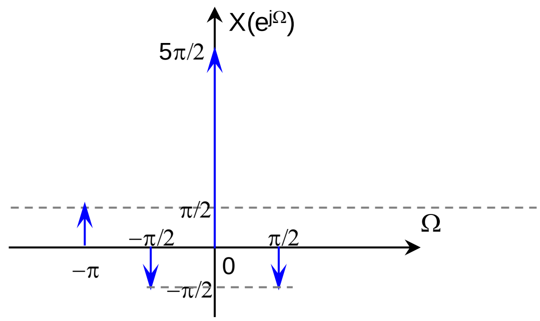
$X[0] = \frac{1}{4}(2e^{-0} + 2Cos(0) + 1) = \frac{5}{4}, \quad X[1] = \frac{1}{4}\left(2e^{j\pi} + 2Cos\left(\frac{\pi}{2}\right) + 1\right) = -\frac{1}{4}$

Finally, over one period:

$$X(e^{j\Omega}) = 2\pi \sum_{k=-2}^{2} X[k]\delta\left(\Omega - k\frac{\pi}{2}\right), -2 \le k \le 1, \quad -\pi \le k\frac{\pi}{2} \le \frac{\pi}{2}$$

$$X(e^{j\Omega}) = 2\pi\left\{\frac{1}{4}\delta(\Omega + \pi) - \frac{1}{4}\delta\left(\Omega + \frac{\pi}{2}\right) + \frac{5}{4}\delta(\Omega) - \frac{1}{4}\delta\left(\Omega - \frac{\pi}{2}\right)\right\}$$





PROBLEM 2 (12 PTS)

Determine the Laplace transform (or Z transform) and the ROC for each of the following signals:

a) $x(t) = e^{at}u(t - k)$.
Use the definition of the Laplace Transform. Sketch the pole-zero plot. What is the condition on a for the Fourier Transform to exist?

b) $x(t) = te^{at}u(t) + e^{at}u(t - 4) + u(t - 5) + 3\delta(t)$.

c) $x[n] = a^{|n|} + \delta[n]$. What is the condition on a for the Z-transform to exist?

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a) $x(t) = e^{at}u(t - k)$

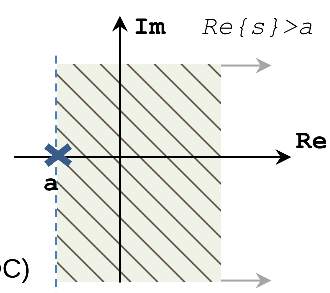
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{k}^{\infty} e^{at}e^{-st}dt = \int_{k}^{\infty} e^{-(s-a)t}dt = -\frac{1}{s-a}e^{-(\sigma+j\omega-a)t}\Big|_{k}^{\infty}$$

$$X(s) = -\frac{1}{s-a}\left(e^{-(\sigma+j\omega-a)\infty} - e^{-(\sigma+j\omega-a)k}\right), \quad e^{(j\omega)(\pm\infty)} \le 1$$

$X(s)$ is integrable if $\sigma - a > 0 \rightarrow Re\{s\} > a$

$$\rightarrow X(s) = \frac{e^{ak}}{s-a}e^{-sk}, Re\{s\} > a$$

For the FT to exist, we need $a < 0$ (so that the $j\omega$ axis is included in the ROC)



b) $x(t) = te^{at}u(t) + e^{at}u(t - 4) + u(t - 5) + 3\delta(t)$

We know:

$$e^{at}u(t) \xleftrightarrow{L} \frac{1}{s-a}, ROC: Re\{s\} > a \quad \Rightarrow \quad te^{at}u(t) \xleftrightarrow{L} \frac{1}{(s-a)^2}, ROC: Re\{s\} > a$$

$$e^{at}u(t-4) \xleftrightarrow{L} \frac{e^{4a}}{s-a}e^{-4s}, ROC: Re\{s\} > a$$

$$u(t-5) \xleftrightarrow{L} \frac{1}{s}e^{-5s}, Re\{s\} > 0 \quad \Big| \quad \delta(t) \xleftrightarrow{L} 1, ROC: \forall s$$

$$\therefore X(s) = \frac{1}{(s-a)^2} + \frac{e^{4a}}{s-a}e^{-4s} + \frac{1}{s}e^{-5s} + 3, \quad Re\{s\} > a \; for \; a > 0, \quad Re\{s\} > 0 \; for \; a < 0$$

c) $x[n] = a^{|n|} + \delta[n] = a^n u[n] + a^{-n}u[-n-1] + \delta[n] = a^n u[n] + (a^{-1})^n u[-n-1] + \delta[n]$

We know:

$$a^n u[n] \xleftrightarrow{Z} \frac{z}{z-a}, ROC: |z| > |a| \quad \Big| \quad (a^{-1})^n u[-n-1] \xleftrightarrow{Z} -\frac{z}{z-\frac{1}{a}}, ROC: |z| < \left|\frac{1}{a}\right|$$

$$\therefore X(z) = 1 + \frac{z}{z-a} - \frac{z}{z-\frac{1}{a}}, \quad |a| < |z| < \left|\frac{1}{a}\right|$$

For the Z-Transform to exist, we need $|a| < 1$.

PROBLEM 3 (10 PTS)

For an LTI system, we are given the Z-transform of the input and output signals:

$$X(z) = \frac{1}{1+z^{-2}+z^{-4}}, \qquad Y(z) = \frac{1}{2}+\frac{1}{4}z^{-1}$$

a) Determine the impulse response $h[n]$. Sketch it. (7)
b) Determine the DTFT of $h[n]$. (2)
c) What is the region of convergence of $H(z)$? Sketch the DTFT on the z-plane. (1)

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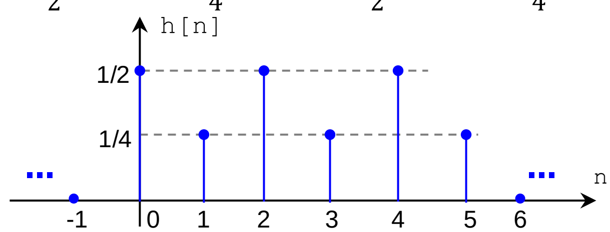
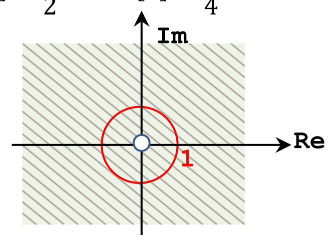
a) $H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}+\frac{1}{4}z^{-1}}{\frac{1}{1+z^{-2}+z^{-4}}} = \left(\frac{1}{2}+\frac{1}{4}z^{-1}\right)(1+z^{-2}+z^{-4}) = \frac{1}{2}+\frac{1}{2}z^{-2}+\frac{1}{2}z^{-4}+\frac{1}{4}z^{-1}+\frac{1}{4}z^{-3}+\frac{1}{4}z^{-5}$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{2}+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}+\frac{1}{4}z^{-3}+\frac{1}{2}z^{-4}+\frac{1}{4}z^{-5}$$

$\rightarrow h[0] = \frac{1}{2}, \quad h[1] = \frac{1}{4}, \quad h[2] = \frac{1}{2}, \quad h[3] = \frac{1}{4}, \quad h[4] = \frac{1}{2}, \quad h[5] = \frac{1}{4}$





b) $H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = \frac{1}{2}+\frac{1}{4}e^{-j\Omega}+\frac{1}{2}e^{-2j\Omega}+\frac{1}{4}e^{-3j\Omega}+\frac{1}{2}e^{-4j\Omega}+\frac{1}{4}e^{-5j\Omega}$
c) $H(z)$ is undefined if $z = 0$. Thus, the ROC is the entire z-plane except for $|z| = 0$.

PROBLEM 4 (15 PTS)

a) For the following Laplace Transform, determine all possible time-domain signals. Hint: First determine all possible ROCs. (8)

$$X(s) = \frac{s}{s^2+(a+b)s+ab}, a, b \in \Re$$

b) For the following Z-transform, determine $h[n]$ (for $a > 1$ and $a < 1$) so that the DTFT exists. (7)

$$H(z) = \frac{z}{z-a}+\frac{z}{z-a^2}$$

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a)

$$X(s) = \frac{s}{s^2+(a+b)s+ab} = \frac{A}{s+a}+\frac{B}{s+b}, A = \frac{a}{a-b}, B = -\frac{b}{a-b}$$

We know:

$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, Re\{s\} > -a$ or $-e^{-at}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, Re\{s\} < -a$

$e^{-bt}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+b}, Re\{s\} > -b$ or $-e^{-bt}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s+b}, Re\{s\} < -b$

Assuming $a > b$: Three possible ROCs:

$Re\{s\} > -a \ \cap \ Re\{s\} > -b \rightarrow x(t) = Ae^{-at}u(t) + Be^{-bt}u(t)$
$Re\{s\} < -a \ \cap \ Re\{s\} < -b \rightarrow x(t) = -Ae^{-at}u(-t) - Be^{-bt}u(-t)$
$Re\{s\} > -a \ \cap \ Re\{s\} < -b \rightarrow x(t) = Ae^{-at}u(t) - Be^{-bt}u(-t)$

b) $H(z) = \frac{z}{z-a}+\frac{z}{z-a^2}$

We know:

$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-a}, ROC: |z| > |a|$ | $-a^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{z-a}, ROC: |z| < |a|$

Possible ROCs for $H(z)$:

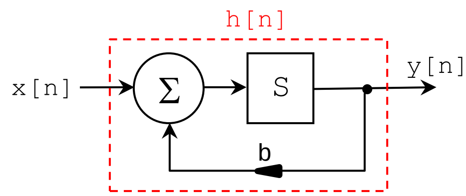
$\|z\| > \|a\| \cap \|z\| > \|a^2\|$	$\|z\| < \|a\| \cap \|z\| < \|a^2\|$	$\|z\| > \|a\| \cap \|z\| < \|a^2\|$	$\|z\| < \|a\| \cap \|z\| > \|a^2\|$

For $a > 1$: We need the unit circle inside the ROC: $\rightarrow \{|z| < |a| \cap |z| < |a^2|\} \equiv |z| < |a|$
Thus: $h[n] = -a^n u[-n-1] + -a^{2n} u[-n-1]$

For $a < 1$: We need the unit circle inside the ROC: $\rightarrow \{|z| > |a| \cap |z| > |a^2|\} \equiv |z| > |a|$
Thus: $h[n] = a^n u[n] + a^{2n} u[n]$

PROBLEM 5 (16 PTS)

For the following LTI system with $h[n]$ causal.



a) Find the difference equation that relates $y[n]$ and $x[n]$. (2)
b) Determine $H(z)$, the ROC, and the pole-zero plot. (3)
c) Determine the impulse response $h[n]$. (3)
d) For the input $x[n] = a^n u[-n]$: (8)
   i. Determine $X(z)$, the region of convergence, and the pole-zero plot.
   ii. Determine $y[n]$. What is the condition on a and b so that the DTFT of $y[n]$ exists?

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a) $y[n] = x[n-1] + by[n-1]$

b)

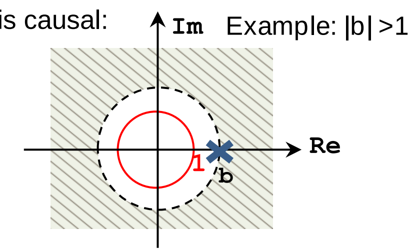
$$Y(z) = z^{-1}X(z) + bz^{-1}Y(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - bz^{-1}} = \frac{1}{z-b}$$

Here, we use a well-know transform along with the fact that $h[n]$ is causal:

$$b^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z-b}, ROC: |z| > |b|$$

$$\rightarrow b^{n-1}u[n-1] \stackrel{z}{\longleftrightarrow} \frac{zz^{-1}}{z-b} = \frac{1}{z-b}, ROC: |z| > |b|$$

$$\rightarrow H(z) = \frac{1}{z-b}, ROC: |z| > |b|$$



c) $h[n] = b^{n-1}u[n-1]$

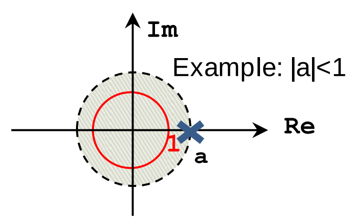
d) $x[n] = a^n u[-n]$

$$a^n u[-n-1] \stackrel{z}{\longleftrightarrow} -\frac{z}{z-a}, ROC: |z| < |a|$$

$$\rightarrow a^{n-1}u[-(n-1)-1] \stackrel{z}{\longleftrightarrow} -\frac{zz^{-1}}{z-a} = \frac{1}{z-a}, ROC: |z| < |a|$$

$$\rightarrow aa^{n-1}u[-(n-1)-1] = a^n u[-n] \stackrel{z}{\longleftrightarrow} -a\frac{1}{z-a}, ROC: |z| < |a|$$

$$\Rightarrow X(z) = -\frac{a}{z-a}, ROC: |z| < |a|$$



Now, to get $y[n]$:

$$Y(z) = H(z)X(z) = -\frac{a}{(z-b)(z-a)} = \frac{A}{z-b} + \frac{B}{z-a}, |b| < |z| < |a|$$

$$\rightarrow A = \frac{a}{b-a}, B = \frac{a}{a-b}$$

We know:

$$-a^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{z}{z-a}, ROC: |z| < |a|$$

$$\rightarrow a^{n-1}u[-(n-1)+1] \stackrel{z}{\longleftrightarrow} \frac{zz^{-1}}{z-a} = \frac{1}{z-a}, ROC: |z| < |a|$$

$$b^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z-b}, ROC: |z| > |b|$$

$$\rightarrow b^{n-1}u[n-1] \stackrel{z}{\longleftrightarrow} \frac{zz^{-1}}{z-b} = \frac{1}{z-b}, ROC: |z| > |b|$$

$$\therefore y[n] = \frac{a}{b-a}b^{n-1}u[n-1] - \frac{a}{a-b}a^{n-1}u[-n] = \frac{a}{b-a}\left(b^{n-1}u[n-1] + a^{n-1}u[-n]\right)$$

For the DTFT of $y[n]$ to exist, the unit circle needs to be inside the ROC. Thus, we need: $|b| < 1 < |a|$

PROBLEM 6 (10 PTS)

a) For an LTI system with $H(s) = \frac{s+5}{s}$. Is the system stable?

b) An LTI system has $H(s),\ a < Re\{s\} < a^2$. What is the condition on a so that the system is stable?

c) For an LTI system with $H(s) = \frac{1}{s+2} + \frac{3}{s-4}$:

   i. If the system is causal, determine the ROC. Does the FT of $h(t)$ exist? Justify.

   ii. If the system is stable, determine the ROC. Is the system causal? Justify.

d) $H(z) = \frac{z(z+1)}{(z-2)\left(z-\frac{1}{2}\right)}$. If $h[n]$ is causal, is the system stable? Justify.

e) Given that $y[n] = 2u[n-3], x[n] = \delta[n] + \delta[n-1]$. Is the system stable? Justify.

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a) The system has a pole at $s = 0$. As a result, the $j\omega$ axis (which is the FT) is not included. As the existence of the FT implies stability, the system is not stable.

b) We need the ROC to include the $j\omega$ axis. For $0 < a < 1$, the ROC does not exist. For $a > 1$, the $j\omega$ axis is not included. For $a < 0$, the $j\omega$ axis is always included. Thus, we need $a < 0$.

c) $H(s) = \frac{1}{s+2} + \frac{3}{s-4}$: There are three possible ROCs. $Re\{s\} > 4$, $Re\{s\} < -2$, $-2 < Re\{s\} < 4$

   i. If the system is causal, the ROC is to the right of the rightmost pole, i.e., $Re\{s\} > 4$. The $j\omega$ axis is not included. Thus, the FT of $h(t)$ does not exist.

   ii. If the system is stable, the ROC has to include the $j\omega$ axis. Thus: $-2 < Re\{s\} < 4$. Since the ROC is not to the right of the rightmost pole, the system is not causal.

d) If the system is causal, then the ROC is outside of the outermost pole, i.e., $z = 2$. In this case, the unit circle is not included in the ROC, and thus the system is not stable.

e) $Y(z) = \frac{2zz^{-3}}{z-1} = \frac{2z^{-2}}{z-1}, X(z) = 1 + z^{-1} = \frac{z+1}{z}, \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-2}}{(z-1)}\frac{z}{(z+1)}$. There is a pole at $z = 1$. Thus, the system is not stable.

PROBLEM 7 (12 PTS)

a) For the following signal: $x(t) = Cos\left(\frac{\pi}{4}t\right) + 2Cos\left(\frac{\pi}{2}t\right) + 3$ (6)

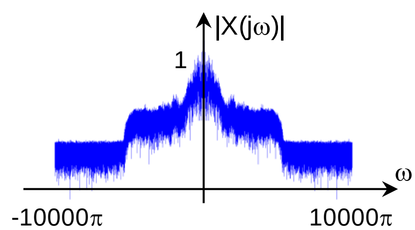
   i. Determine the Fourier Series and the fundamental angular frequency. Determine the Fourier Transform and sketch it.

   ii. Sketch the FT of the sampled signal for $T_S = 4,\ T_S = 2\ secs$. Indicate the cases that result in aliasing. What is the condition on T_S to avoid aliasing?

b) For the following speech signal: (6)

   i. Sketch the FT of the sampled signal for Ts = 0.05, 0.1, and 0.2ms. What is the condition on T_S to avoid aliasing?

   ii. We want to transmit this signal over a channel that only allows frequencies from 200 Hz to 3.2KHz. Sketch the FT of the system that allows proper transmission of the given speech signal. For the new band-limited signal, what is the condition on T_S that avoids aliasing when sampling?



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a) $x(t) = Cos\left(\frac{\pi}{4}t\right) + 2Cos\left(\frac{\pi}{2}t\right) + 3$

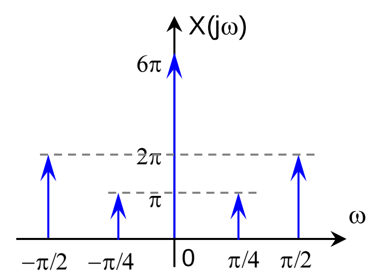
$\frac{\pi}{4}T = 2\pi r, \frac{\pi}{2}T = 2\pi k \rightarrow\ T = 8r = 4k \rightarrow r = 1, k = 2 \Rightarrow T = 8, \omega_0 = \frac{\pi}{4}$

$x(t) = \frac{1}{2}e^{j\frac{\pi t}{4}} + \frac{1}{2}e^{-\frac{\pi t}{4}} + e^{\frac{\pi t}{2}} + e^{-\frac{\pi t}{2}}$

$x(t) = X[1]e^{j\pi t} + X[-1]e^{-j\pi t} + X[2]e^{j2\pi t} + X[-2]e^{-j2\pi t} + X[0]$

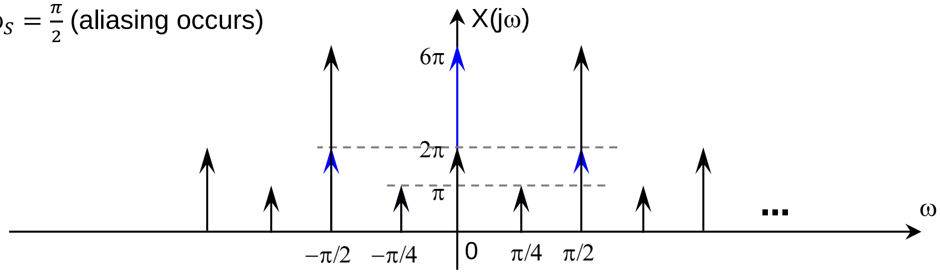
Thus:

$$X[k] = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 1, & k = \pm 2 \\ 3, & k = 0 \\ 0, & otherwise \end{cases}$$

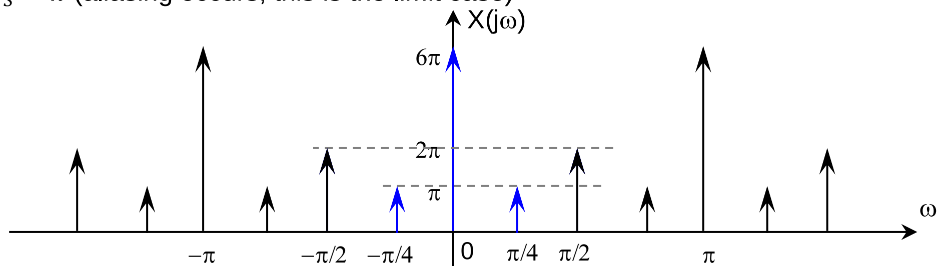


$$\therefore X(j\omega) = \pi\delta\left(\omega - \frac{\pi}{4}\right) + \pi\delta\left(\omega + \frac{\pi}{4}\right) + 2\pi\delta\left(\omega - \frac{\pi}{2}\right) + 2\pi\delta\left(\omega + \frac{\pi}{2}\right) + 6\pi\delta(\omega)$$

$T_S = 4 \rightarrow \omega_S = \frac{\pi}{2}$ (aliasing occurs)

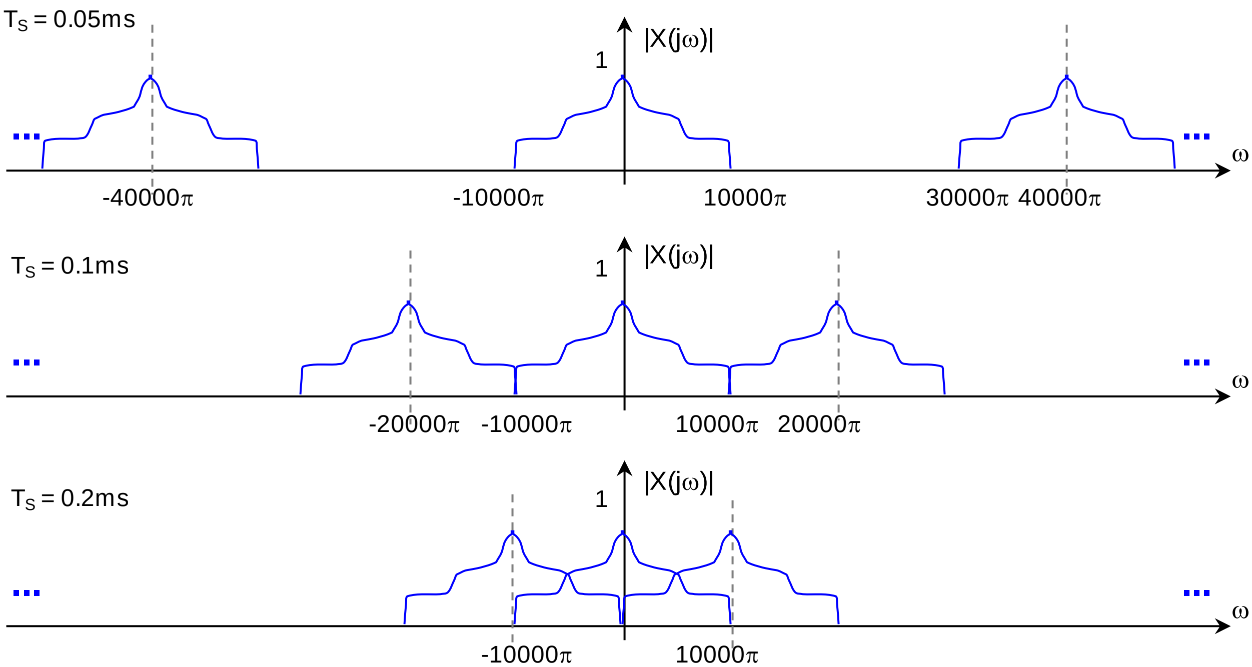


$T_S = 2 \rightarrow \omega_S = \pi$ (aliasing occurs, this is the limit case)



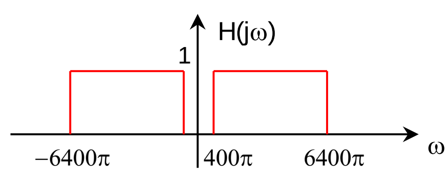
To avoid aliasing: $\omega_S > 2\omega_m \rightarrow \omega_S > 2\frac{\pi}{2} \rightarrow \frac{2\pi}{T_S} > \pi \rightarrow Ts < 2\ secs$

b) $T_S = 0.05ms \rightarrow \omega_S = 40000\pi$, $T_S = 0.1ms \rightarrow \omega_S = 20000\pi$, $T_S = 0.2ms \rightarrow \omega_S = 10000\pi$



To avoid aliasing we need: $\omega_S > 2\omega_m \rightarrow \omega_S > 2(10000\pi) \rightarrow \frac{2\pi}{T_S} > 20000\pi \rightarrow Ts < 0.1ms$

To allow transmission over a band of 200 - 3200Hz, i.e. from $400\pi\ to\ 6400\pi$, we need the following filter:



The new $\omega_m = 6400\pi$. Thus, the condition to avoid aliasing when sampling is:

$$\omega_S > 2\omega_m \rightarrow \omega_S > 2(6400\pi) \rightarrow \frac{2\pi}{T_S} > 12800\pi \rightarrow Ts < 0.15625ms$$

PROBLEM 8 (10 PTS)

c) For the following signal: $m(t) = 3Cos\left(\frac{\pi}{6}t\right) - 1$ and for Full Amplitude modulation: (6)

i. Get the Fourier series and the Fourier Transform of $m(t)$. Sketch the Fourier Transform of $m(t)$ (the message) and of the modulated signal.

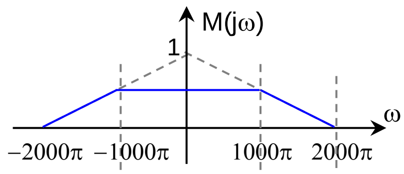
ii. Provide an expression for the percentage of modulation based on k_a. What is condition on k_a to avoid over-modulation? Find the value of k_a that provides a % of modulation of 45%.

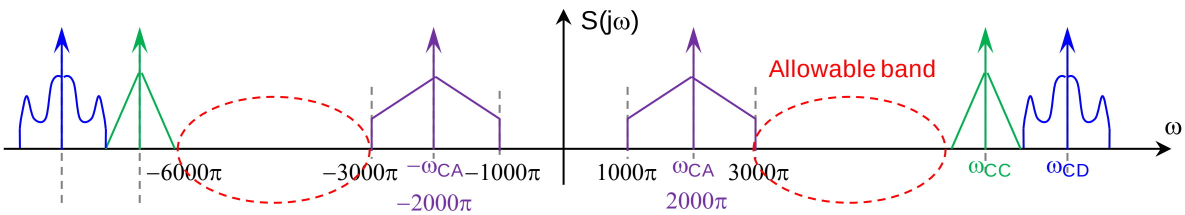
d) The following signal needs to be transmitted alongside others (Full AM). However, there is only an allowable band for the signal. Thus, we need to filter some frequencies. (4)

i. Sketch the FT of the system that filters these frequencies.

ii. What is the carrier frequency (in Hz) that allows for this signal to be properly transmitted?

iii. What is the condition on k_a to avoid over-modulation?





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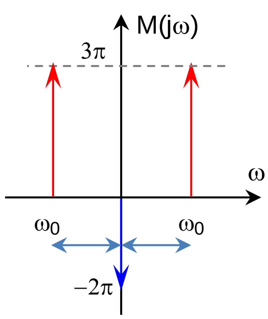
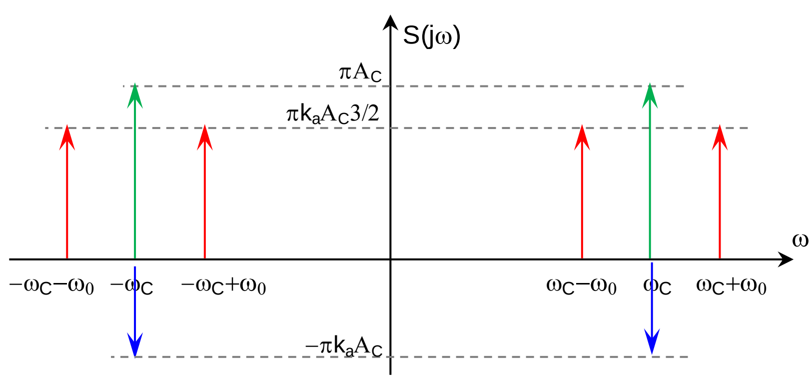
a) $m(t) = 3Cos\left(\frac{\pi}{6}t\right) - 1 = \frac{3}{2}e^{j\frac{\pi}{6}t} + \frac{3}{2}e^{-j\frac{\pi}{6}t} - 1 = M[1]e^{j\frac{\pi}{6}t} + M[-1]e^{-j\frac{\pi}{6}t} + M[0], \omega_0 = \frac{\pi}{6}$

$$M[k] = \begin{cases} \frac{3}{2}, & k = \pm 1 \\ -1, & k = 0 \\ 0, & otherwise \end{cases}$$

$$M(j\omega) = 2\pi\left\{\frac{3}{2}\delta\left(\omega - \frac{\pi}{6}\right) + \frac{3}{2}\delta\left(\omega + \frac{\pi}{6}\right) - \delta(\omega)\right\}$$



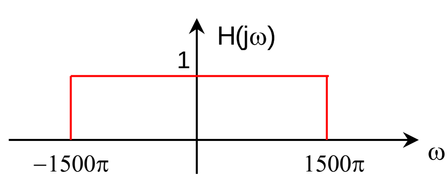


Percentage of modulation: $max|k_a m(t)| = |k_a| max|m(t)| = |k_a|4$

To avoid over-modulation: $|k_a|4 < 1 \rightarrow |k_a| < \frac{1}{4}$

For 45%: $|k_a|4 = 0.45 \rightarrow |k_a| = 0.45/4$

b) Allowable band: 3000π. So, $\omega_m = 1500\pi$.
Then, we need the following filter:



According to the graph, the carrier frequency should be: $\omega_c = 4500\pi \rightarrow f_c = 2250Hz$.
To avoid over-modulation: $max|k_a m(t)| < 1$

BONUS PROBLEM (+ 10 PTS)

a) Using the definition, determine the Laplace Transform, the ROC and the pole-zero plot of the following signal: $x(t) = e^{at}(u(t-T) - u(t))$. (2)

b) Demonstrate that if the ROC of the Z-transform of the impulse response of an LTI system contains the unit circle, then, the system is stable. (2)

c) Consider a signal with Laplace Transform $X(s) = \frac{1}{s+\frac{1}{2}}$. Determine a different Laplace Transform $X_1(s)$ and the corresponding time function $x_1(t)$, so that: (2)

$$|X_1(j\omega)| = |X(j\omega)|, \qquad but\ \ x_1(t) \neq x(t)$$

d) A pressure gauge, which can be modeled as an LTI system, has a time response to a unit impulse input given by $(1 + e^{-t} - 6te^{-t})u(t)$. For a certain unknown input $x(t)$, the output is observed to be $(2 - 2e^{-t} + te^{-t})u(t)$. For this observed measurement, determine the true pressure input to the gauge as a function of time. Assume that the input signal is right-sided. (4)

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a) T > 0:

$$X(s) = -\int_0^T e^{at}e^{-st}dt = \frac{1}{s-a}e^{-(s-a)t}\Big|_0^T$$

$$X(s) = \frac{1}{s-a}\left(e^{-(s-a)T} - 1\right), s \neq a$$

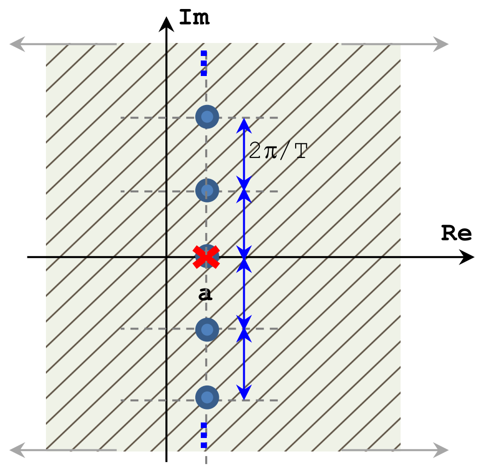
$$X(s) = \int_0^T e^{at}e^{-st}dt = \int_0^T 1dt = T, s = a$$

Pole: $s = a$

Zeros: $\left(e^{-(s-a)T} - 1\right) = 0 \rightarrow -(s-a)T = j2\pi k$

$$\rightarrow s = a - \frac{j2\pi k}{T}, k = 0, \pm 1, \pm 2, \ldots$$

There is one zero at $s = a$. As a result, there is pole-zero cancellation. Thus, the ROC is the entire s-plane.



b) If the ROC includes the unit circle, that means that the DTFT exists or converges.
Convergence of DTFT: $\sum_{n=-\infty}^{\infty}|h[n]| < \infty$
Definition of stability for LTI systems: $\sum_{n=-\infty}^{\infty}|h[n]| < \infty$
We notice that these two definitions are the same, therefore convergence of the DTFT of $h[n]$ implies stability of the LTI system.

c) $X(s) = \frac{1}{s+\frac{1}{2}} \rightarrow X(j\omega) = \frac{1}{j\omega+\frac{1}{2}} = \frac{j\omega-\frac{1}{2}}{\omega^2+\frac{1}{4}} \rightarrow |X(j\omega)| = \frac{\sqrt{\omega^2+\frac{1}{2}}}{\omega^2+\frac{1}{4}}$

$\rightarrow |X_1(j\omega)| = \frac{\sqrt{\omega^2+\frac{1}{2}}}{\omega^2+\frac{1}{4}}$. This can correspond to: $X_1(j\omega) = \frac{j\omega+\frac{1}{2}}{\omega^2+\frac{1}{4}} = \frac{1}{j\omega-\frac{1}{2}} \rightarrow X_1(s) = \frac{1}{s-\frac{1}{2}}$

Now, we know that $X_1 = \frac{1}{s-\frac{1}{2}}, ROC: Re\{s\} > \frac{1}{2}\ or\ Re\{s\} < \frac{1}{2}$. For FT to exist, we need $Re\{s\} < \frac{1}{2}$

Thus, $x_1(t) = -e^{\frac{1}{2}t}u(-t)$

d) $H(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{6}{(s+1)^2}, Y(s) = \frac{2}{s} - \frac{2}{s+1} + \frac{1}{(s+1)^2}$

$$\rightarrow X(s) = \frac{Y(s)}{H(s)} = \frac{\frac{2}{s} - \frac{2}{s+1} + \frac{1}{(s+1)^2}}{\frac{1}{s} + \frac{1}{s+1} - \frac{6}{(s+1)^2}} = \frac{2(s+1)^2 - 2s(s+1) + s}{(s+1)^2 + s(s+1) - 6s} = \frac{2s^2 + 4s + 2 - 2s^2 - 2s + s}{s^2 + 2s7 + 1 + s^2 + s - 6s}$$

$$\rightarrow X(s) = \frac{3s+2}{2s^2 - 3s + 1} = \frac{3s+2}{2(s-1)\left(s-\frac{1}{2}\right)} = \frac{5}{s-1} - \frac{7}{2\left(s-\frac{1}{2}\right)}$$

Since we know that $x(t)$ is right-sided, then the ROC has to extend toward infinity.

$$e^t u(t) \overset{L}{\longleftrightarrow} X(s) = \frac{1}{s-1}, ROC: Re\{s\} > 1 \qquad\Big|\qquad e^{\frac{1}{2}t}u(t) \overset{L}{\longleftrightarrow} X(s) = \frac{1}{s-\frac{1}{2}}, ROC: Re\{s\} > \frac{1}{2}$$

Finally: $x(t) = 5e^t u(t) - \frac{7}{2}e^{\frac{1}{2}t}u(t)$