Final Exam

(July 25th: 2pm to 4pm)

Show your procedure in detail!

PROBLEM 1 (15 PTS)

a) For the following Fourier Transform of a periodic signal: Determine the fundamental angular frequency and the Fourier series coefficients. Then determine the corresponding time-domain signal.

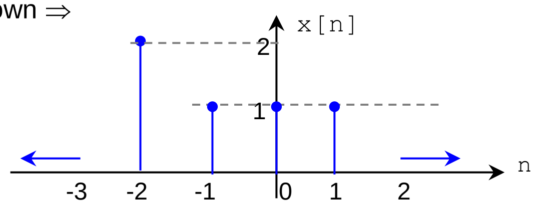
$$X(j\omega) = j\delta\left(\omega - \frac{\pi}{3}\right) + 2\delta\left(\omega - \frac{\pi}{7}\right)$$

b) Determine the Fourier Series, the fundamental angular frequency, and the Fourier Transform of the following signal:

$$x(t) = Cos(\pi t) + 3jSin(2\pi t)$$

c) Determine the Discrete Time Fourier Series and the fundamental angular frequency of the following signal (sketch one period of $X[k]$). Then, determine the DTFT for one period (provide the equation and sketch it).

i. $x[n]$ with one period shown ⇒



PROBLEM 2 (12 PTS)

Determine the Laplace transform (or Z transform) and the ROC for each of the following signals:

a) $x(t) = e^{at}u(t-k)$.
Use the definition of the Laplace Transform. Sketch the pole-zero plot. What is the condition on a for the Fourier Transform to exist?

b) $x(t) = te^{at}u(t) + e^{at}u(t-4) + u(t-5) + 3\delta(t)$.

c) $x[n] = a^{|n|} + \delta[n]$. What is the condition on a for the Z-transform to exist?

PROBLEM 3 (10 PTS)

For an LTI system, we are given the Z-transform of the input and output signals:

$$X(z) = \frac{1}{1 + z^{-2} + z^{-4}}, \qquad Y(z) = \frac{1}{2} + \frac{1}{4}z^{-1}$$

a) Determine the impulse response $h[n]$. Sketch it. (7)
b) Determine the DTFT of $h[n]$. (2)
c) What is the region of convergence of $H(z)$? Sketch the DTFT on the z-plane. (1)

PROBLEM 4 (15 PTS)

a) For the following Laplace Transform, determine all possible time-domain signals. Hint: First determine all possible ROCs. (8)

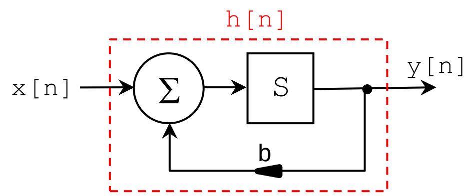
$$X(s) = \frac{s}{s^2 + (a+b)s + ab}, a, b \in \Re$$

b) For the following Z-transform, determine $h[n]$ (for $a > 1$ and $a < 1$) so that the DTFT exists. (7)

$$H(z) = \frac{z}{z-a} + \frac{z}{z-a^2}$$

PROBLEM 5 (16 PTS)

For the following LTI system with $h[n]$ causal.



a) Find the difference equation that relates $y[n]$ and $x[n]$. (2)
b) Determine $H(z)$, the ROC, and the pole-zero plot. (3)
c) Determine the impulse response $h[n]$. (3)
d) For the input $x[n] = a^n u[-n]$: (8)
  i. Determine $X(z)$, the region of convergence, and the pole-zero plot.
  ii. Determine $y[n]$. What is the condition on a and b so that the DTFT of $y[n]$ exists?

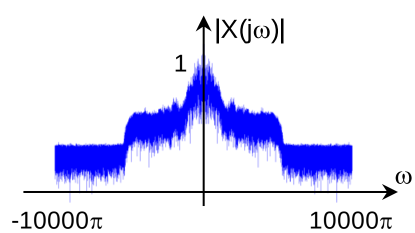
PROBLEM 6 (10 PTS)

a) For an LTI system with $H(s) = \frac{s+5}{s}$. Is the system stable?
b) An LTI system has $H(s)$, $a < Re\{s\} < a^2$. What is the condition on a so that the system is stable?
c) For an LTI system with $H(s) = \frac{1}{s+2} + \frac{3}{s-4}$:
  i. If the system is causal, determine the ROC. Does the FT of $h(t)$ exist? Justify.
  ii. If the system is stable, determine the ROC. Is the system causal? Justify.
d) $H(z) = \frac{z(z+1)}{(z-2)\left(z-\frac{1}{2}\right)}$. If $h[n]$ is causal, is the system stable? Justify.
e) Given that $y[n] = 2u[n-3], x[n] = \delta[n] + \delta[n-1]$. Is the system stable? Justify.

PROBLEM 7 (12 PTS)

a) For the following signal: $x(t) = Cos\left(\frac{\pi}{4}t\right) + 2Cos\left(\frac{\pi}{2}t\right) + 3$ (6)
  i. Determine the Fourier Series and the fundamental angular frequency. Determine the Fourier Transform and sketch it.
  ii. Sketch the FT of the sampled signal for $T_S = 4$, $T_S = 2\ secs$. Indicate the cases that result in aliasing. What is the condition on T_S to avoid aliasing?

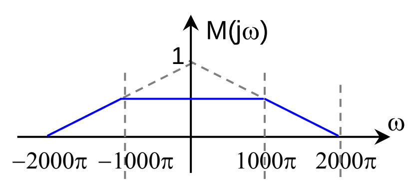
b) For the following speech signal: (6)
  i. Sketch the FT of the sampled signal for Ts = 0.05, 0.1, and 0.2ms. What is the condition on T_S to avoid aliasing?
  ii. We want to transmit this signal over a channel that only allows frequencies from 200 Hz to 3.2KHz. Sketch the FT of the system that allows proper transmission of the given speech signal. For the new band-limited signal, what is the condition on T_S that avoids aliasing when sampling?

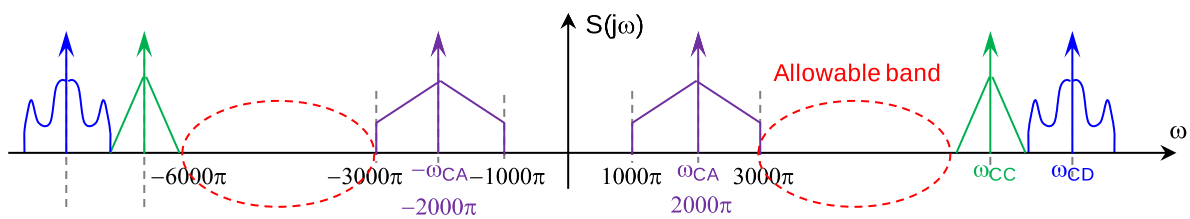


PROBLEM 8 (10 PTS)

a) For the following signal: $m(t) = 3Cos\left(\frac{\pi}{6}t\right) - 1$ and for Full Amplitude modulation: (6)
  i. Get the Fourier series and the Fourier Transform of $m(t)$. Sketch the Fourier Transform of $m(t)$ (the message) and of the modulated signal.
  ii. Provide an expression for the percentage of modulation based on k_a. What is condition on k_a to avoid over-modulation? Find the value of k_a that provides a % of modulation of 45%.

b) The following signal needs to be transmitted alongside others (Full AM). However, there is only an allowable band for the signal. Thus, we need to filter some frequencies. (4)
  i. Sketch the FT of the system that filters these frequencies.
  ii. What is the carrier frequency (in Hz) that allows for this signal to be properly transmitted?
  iii. What is the condition on k_a to avoid over-modulation?





BONUS PROBLEM (+ 10 PTS)

a) Using the definition, determine the Laplace Transform, the ROC and the pole-zero plot of the following signal: $x(t) = e^{at}(u(t-T) - u(t))$. (2)

b) Demonstrate that if the ROC of the Z-transform of the impulse response of an LTI system contains the unit circle, then, the system is stable. (2)

c) Consider a signal with Laplace Transform $X(s) = \frac{1}{s+\frac{1}{2}}$. Determine a different Laplace Transform $X_1(s)$ and the corresponding time function $x_1(t)$, so that: (2)

$$|X_1(j\omega)| = |X(j\omega)|, \quad but\ x_1(t) \neq x(t)$$

d) A pressure gauge, which can be modeled as an LTI system, has a time response to a unit impulse input given by $(1 + e^{-t} - 6te^{-t})u(t)$. For a certain unknown input $x(t)$, the output is observed to be $(2 - 2e^{-t} + te^{-t})u(t)$. For this observed measurement, determine the true pressure input to the gauge as a function of time. Assume that the input signal is right-sided. (4)

USEFUL FORMULAS

Time Domain	Periodic (t,n)	Non-periodic (t,n)	
CT	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$ $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $x(t)$ has period T, $\omega_0 = 2\pi/T$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	
DT	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$ $x[n], X[k]$ have period N, $\Omega_0 = 2\pi/N$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	
	Discrete (k)	Continuous (ω, Ω)	Frequency Domain

$$\sum_{n=0}^{M-1} b^n = \frac{1-b^M}{1-b}, b \neq 1 \qquad \sum_{n=0}^{\infty} b^n = \frac{1}{1-b}, |b| < 1$$

$$\sum_{n=k}^{l} b^n = \frac{b^k - b^{l+1}}{1-b}, b \neq 1 \qquad \sum_{n=k}^{\infty} b^n = \frac{b^k}{1-b}, |b| < 1$$

$$cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

FT of a periodic signal:

$$x(t) \overset{FT}{\leftrightarrow} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

DTFT of a periodic signal (for one period):

$$x[n] = \sum_{N} X[k]e^{jk\Omega_0 n} \overset{DTFT}{\longleftrightarrow} X(e^{j\Omega}) = 2\pi \sum_{N} X[k]\delta(\Omega - k\Omega_0)$$

Full AM modulation:

$$s(t) = (1 + k_a m(t))A_C Cos(\omega_C t)$$
$$S(j\omega) = \pi A_C[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)] + \frac{k_a A_C}{2}\left[M\big(j(\omega + \omega_C)\big) + M\big(j(\omega - \omega_C)\big)\right]$$

Properties of the Laplace Transform:

- Time-shift:

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad ROC: R$$
$$\rightarrow x(t - t_0) \overset{L}{\longleftrightarrow} e^{-st_0}X(s), ROC: R$$

- Time differentiation:

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad ROC: R$$
$$\rightarrow \frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s), ROC: includes\ R$$

- Differentiation in the s-domain

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad ROC: R$$
$$\rightarrow -tx(t) \overset{L}{\longleftrightarrow} \frac{dX(s)}{ds}, ROC: R$$

Properties of Z-Transform:

- Time-shift:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow x[n - k] \overset{Z}{\longleftrightarrow} z^{-k}X(z), ROC: R_x\ except\ possibly\ the\ addition\ or\ deletion\ of\ 0\ or\ \infty$$

- Differentiation in the z-domain

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad ROC: R_x$$