Notes - Class # 12

Z-TRANSFORM DEFINITION

Discrete-Time Fourier Transform:

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Let's replace $z = e^{j\Omega} \rightarrow \ |z| \leq 1$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-\text{n}}$$

- The **Z-Transform** arises when we consider a general number $z = re^{j\Omega} \rightarrow \ |z| \leq r, r\ real$

$$x[n] \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-\text{n}}$$

- Notice that:

$$X(z)|_{z=e^{j\Omega}} = X(e^{j\Omega}): DTFT\ \ of\ x(t)$$

- Also:

$$X(z) = X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\Omega n}$$

We can then say that:

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$$
$$x[n]r^{-n} \xleftrightarrow{DTFT} X(re^{j\Omega}) = X(z)$$

The Z-Transform of $x[n]$ is equivalent to the Discrete-Time Fourier Transform of $x[n]r^{-n}$.

CONVERGENCE OF THE Z-TRANSFORM

- For the Discrete-Time Fourier Transform: $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
  Convergence occurs when: $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$

  Similarly, for the Z-Transform: $x[n]r^{-n} \xleftrightarrow{DTFT} X(re^{j\Omega}) = X(z)$

  Convergence occurs when: $\sum_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty$

- The collection of values of r for which $X(z)$ converges is called the **Region of Convergence (ROC)** of the Z-Transform. The Z-Transform is uniquely defined by $X(z)$ and its Region of Convergence (ROC).
- If ROC includes the unit circle, then the DTFT exists.

Example: Determine the Z-transform of $x[n] = a^n u[n]$.

Let's first determine the Discrete-Time Fourier Transform:

$$x[n] = a^n u[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n$$

Convergence criterion: $\sum_{n=-\infty}^{\infty}|x[n]| < \infty \rightarrow \sum_{n=0}^{\infty}|a^n| < \infty$
This is only possible if $|a| < 1$:

$$\rightarrow X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}}, |a| < 1$$

Now, we determine the Z-Transform:

$$x[n] \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Convergence criterion: $\sum_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty \rightarrow \sum_{n=0}^{\infty}|(ar^{-1})^n| < \infty$

$$\rightarrow |ar^{-1}| < 1 \rightarrow |r| > |a| \rightarrow |z| > |a|$$

$$\rightarrow X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, ROC: |z| > |a|$$

Therefore:

$$x[n] = a^n u[n] \xleftrightarrow{Z} X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, ROC: |z| > |a|$$

Special case:

$$x[n] = u[n] \xleftrightarrow{Z} X(z) = \frac{z}{z-1}, ROC: |z| > 1$$

- The DTFT does not converge for signals of the form $x[n] = a^n u[n], |a| > 1$
- The Z-Transform does converge for $x[n] = a^n u[n]$, $\forall a$, but it imposes a restriction on 'z', i.e. $|z| > |a|$. In other words, the Z-Transform $X(z)$ of $x[n]$ converges for some values of 'z'.

Example: Determine the Z-transform of $x[n] = -a^n u[-n-1]$.

Let's first determine the Discrete-Time Fourier Transform:

$$x[n] = -a^n u[-n-1] \xleftrightarrow{DTFT} X(e^{j\Omega}) = -\sum_{n=-\infty}^{-1} a^n e^{-j\Omega n} = -\sum_{n=-\infty}^{-1} (ae^{-j\Omega})^n = -\sum_{n=1}^{\infty} (ae^{-j\Omega})^{-n}$$

Convergence criterion: $\sum_{n=-\infty}^{\infty}|x[n]| < \infty \rightarrow \sum_{n=-\infty}^{1}|a^n| < \infty, \rightarrow \sum_{n=1}^{\infty}|a^{-n}| < \infty$

This is only possible if $|a| > 1$:

$$\rightarrow X(e^{j\Omega}) = -\sum_{n=1}^{\infty} (a^{-1}e^{j\Omega})^n = 1 - \sum_{n=0}^{\infty} (a^{-1}e^{j\Omega})^n = 1 - \frac{1}{1-a^{-1}e^{j\Omega}} = \frac{1}{1-ae^{-j\Omega}}, |a| < 1$$

Now, we determine the Z-Transform:

$$x[n] \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

Convergence criterion: $\sum_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty \rightarrow \sum_{n=-\infty}^{-1}|(ar^{-1})^n| < \infty, \rightarrow \sum_{n=1}^{\infty}|(ar^{-1})^{-n}| < \infty$

$$\rightarrow |a^{-1}r| < 1 \rightarrow |r| < |a| \rightarrow |z| < |a|$$

$$\rightarrow X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, ROC: |z| < |a|$$

Therefore:

$$x[n] = -a^n u[-n-1] \xleftrightarrow{Z} X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, ROC: |z| < |a|$$

Special case:

$$x[n] = -u[-n-1] \xleftrightarrow{Z} X(z) = \frac{z}{z-1}, ROC: |z| < 1$$

- The DTFT does not converge for signals of the form $x[n] = -a^n u[-n-1], |a| < 1$
- The Z-Transform does converge for $x[n] = -a^n u[-n-1]$, $\forall a$, but it imposes a restriction on 'z', i.e. $|z| < |a|$. In other words, the Z-Transform $X(z)$ of $x[n]$ converges for some values of 'z'.

PROPERTIES OF THE REGION OF CONVERGENCE

1. ROC of $X(z)$: It consists of a ring in the z-plane centered about the origin. If the origin is included, we get disks.
   *Convergence depends only on $|z| = r$, not on Ω

2. ROC of $X(z)$: It does not contain any poles.

3. If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0, \infty$.

4. If $x[n]$ is right-sided and if $|z| = r_0$ is in the ROC, then $|z| > r_0$ is in the ROC.

5. If $x[n]$ is left-sided and if $|z| = r_0$ is in the ROC, then $|z| < r_0$ is in the ROC.

6. If $x[n]$ is two-sided, and if $|z| = r_0$ is in the ROC, then the ROC consists of a ring that includes the circle $|z| = r_0$.

PROPERTIES OF THE Z-TRANSFORM

- Linearity:

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), ROC: R_1$$
$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), ROC: R_2$$
$$\rightarrow ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad ROC: contains\ R_1 \cap R_2$$

- Time-shift:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow x[n-k] \overset{Z}{\longleftrightarrow} z^{-k}X(z), ROC: R_x\ except\ possibly\ the\ addition\ or\ deletion\ of\ 0\ or\ \infty$$

- Frequency-shift (z- shift)

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow e^{j\Omega_0 n}x[n] \overset{Z}{\longleftrightarrow} e^{-j\Omega_0}X(z), \qquad ROC: R_x$$

  In general:

$$z_0^n x[n] \overset{Z}{\longleftrightarrow} e^{-j\Omega_0}X\left(\frac{z}{z_0}\right), \qquad ROC: z_0 R_x, \qquad z_0 = re^{j\Omega_0 n}$$

- Time Reversal:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow x[-n] \overset{Z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \qquad ROC: \frac{1}{R_x} \equiv if z_0 \in\ ROC\ of\ x[n], \rightarrow \frac{1}{z_0} \in ROC\ of\ x[-n]$$

- Convolution

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), ROC: R_1$$
$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), ROC: R_2$$
$$\rightarrow x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)X_2(z), \qquad ROC: contains\ R_1 \cap R_2$$

- Differentiation in the z-domain

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad ROC: R_x$$
$$\rightarrow nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad ROC: R_x$$

- Initial Value Theorem

$$If\ x[n] = 0, n < 0, then:$$
$$x[0] = \lim_{z\to\infty} X(z)$$