Notes - Class # 10

LAPLACE TRANSFORM DEFINITION

Fourier Transform:

$$x(t) \xleftrightarrow{FT} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Let's replace $s = j\omega$ (here, s is purely imaginary).

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- The **Laplace Transform** arises when we consider a general complex number $s = \sigma + j\omega$:

$$x(t) \xleftrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Notice that:

$$X(s)|_{s=j\omega} = X(j\omega): FT \;\; of \; x(t)$$

- Also:

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} \{x(t)e^{-\sigma t}\}e^{-j\omega t}dt$$

We can then say that:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$
$$x(t)e^{-\sigma t} \xleftrightarrow{FT} X(\sigma + j\omega) = X(s)$$

The Laplace Transform of $x(t)$ is equivalent to the Fourier Transform of $x(t)e^{-\sigma t}$.

CONVERGENCE OF THE LAPLACE TRANSFORM

- For the Fourier Transform: $x(t) \xleftrightarrow{FT} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
  Convergence occurs when: $\int_{-\infty}^{\infty}|x(t)|dt < \infty$

- Similarly, for the Laplace Transform: $x(t)e^{-\sigma t} \xleftrightarrow{FT} X(\sigma + j\omega) = X(s)$
  Convergence occurs when: $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$

Example: Determine the Laplace transform of $x(t) = e^{-at}u(t)$.

Let's first determine the Fourier Transform:

$$x(t) = e^{-at}u(t) \xleftrightarrow{FT} X(j\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

Convergence criterion: $\int_{-\infty}^{\infty}|x(t)|dt < \infty \rightarrow \int_{0}^{\infty}|e^{-at}|dt < \infty$
This is only possible if a > 0: $\rightarrow X(j\omega) = -\frac{1}{a+j\omega}\left(e^{-(a+j\omega)\infty} - e^{0}\right) = \frac{1}{a+j\omega}, a > 0$

Now, we determine the Laplace Transform:

$$x(t) = e^{-at}u(t) \xleftrightarrow{L} X(s) = \int_{0}^{\infty} e^{-at}e^{-st}dt = -\frac{1}{a+s}e^{-(a+\sigma+j\omega)t}\Big|_{0}^{\infty}$$

Convergence criterion: $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty \rightarrow \int_{0}^{\infty}\left|e^{-(a+\sigma)t}\right|dt < \infty \Rightarrow a + \sigma > 0$
This is true only if $a + \sigma > 0, or\ Re\{s\} > -a \rightarrow X(s) = -\frac{1}{a+s}\left(e^{-(a+\sigma j\omega)\infty} - e^{0}\right) = \frac{1}{s+a}$

The collection of values of 's' for which $X(s)$ converges is called the **Region of Convergence (ROC)** of the Laplace Transform. The Laplace Transform is uniquely defined by $X(s)$ and its Region of Convergence (ROC). Therefore:

$$x(t) = e^{-at}u(t) \xleftrightarrow{L} X(s) = \frac{1}{s+a}, ROC: Re\{s\} > -a$$

- The Fourier Transform does not converge for signals of the form $x(t) = e^{-at}u(t), a < 0$.
- The Laplace Transform does converge for $x(t) = e^{-at}u(t), \forall a$, but it imposes a restriction on 's', i.e. $Re\{s\} > -a$. In other words, the Laplace Transform $X(s)$ of $x(t)$ converges for some values of 's'.

Example: Determine the Laplace transform of $x(t) = -e^{-at}u(-t)$.

Let's first determine the Fourier Transform:

$$x(t) = -e^{-at}u(-t) \xleftrightarrow{FT} X(j\omega) = -\int_{-\infty}^{0} e^{-at}e^{-j\omega t}dt = \frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{-\infty}^{0}$$

Convergence criterion: $\int_{-\infty}^{\infty}|x(t)|dt < \infty \rightarrow \int_{-\infty}^{0}|e^{-at}|dt < \infty$

This is only possible if a < 0: $\rightarrow X(j\omega) = \frac{1}{a+j\omega}\left(e^{0} - e^{-(a+j\omega)(-\infty)}\right) = \frac{1}{a+j\omega}, a < 0$

Now, we determine the Laplace Transform:

$$x(t) = -e^{-at}u(-t) \xleftrightarrow{L} X(s) = -\int_{-\infty}^{0} e^{-at}e^{-st}dt = \frac{1}{a+s}e^{-(a+\sigma+j\omega)t}\Big|_{-\infty}^{0}$$

Convergence criterion: $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty \rightarrow \int_{-\infty}^{0}\left|e^{-(a+\sigma)t}\right|dt < \infty \Rightarrow a + \sigma < 0$

This is true only if $a + \sigma < 0,\ or\ Re\{s\} < -a \rightarrow X(s) = \frac{1}{a+s}\left(e^{0} - e^{-(a+\sigma j\omega)(-\infty)}\right) = \frac{1}{s+a}$

The collection of values of 's' for which $X(s)$ converges is called the **Region of Convergence (ROC)** of the Laplace Transform. The Laplace Transform is uniquely defined by $X(s)$ and its Region of Convergence (ROC). Therefore:

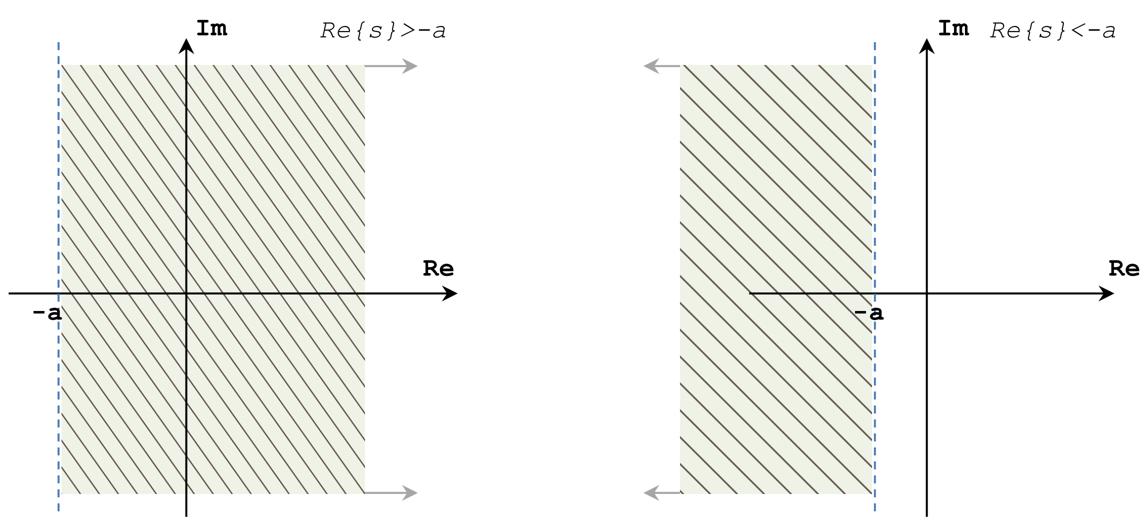
$$x(t) = -e^{-at}u(-t) \xleftrightarrow{L} \frac{1}{s+a}, ROC: Re\{s\} < -a$$

- The Fourier Transform does not converge for signals of the form $x(t) = -e^{-at}u(-t), a > 0$.
- The Laplace Transform does converge for $x(t) = -e^{-at}u(-t), \forall a$, but it imposes a restriction on 's', i.e. $Re\{s\} < -a$. In other words, the Laplace Transform $X(s)$ of $x(t)$ converges for some values of 's'.

In conclusion, the Laplace Transform is specified by $X(s)$ and the ROC:

$$x(t) = e^{-at}u(t) \xleftrightarrow{L} X(s) = \frac{1}{s+a}, ROC: Re\{s\} > -a$$

$$x(t) = -e^{-at}u(-t) \xleftrightarrow{L} X(s) = \frac{1}{s+a}, ROC: Re\{s\} < -a$$



Example: Determine the Laplace transform of $x(t) = e^{-t}u(t) + e^{-4t}u(t)$.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-st}dt + \int_{0}^{\infty} e^{-4t}e^{-st}dt = \frac{1}{s+1} + \frac{1}{s+4}$$

Now, we need to determine the ROC:

$$e^{-t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+1}, ROC: Re\{s\} > -1$$
$$e^{-4t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+4}, ROC: Re\{s\} > -4$$

The region of convergence of $x(t) = e^{-t}u(t) + e^{-4t}u(t)$ has to contain the intersection of the ROCs of the individual terms, i.e.: ROC of $x(t)$: $\{Re\{s\} > -1\} \cap \{Re\{s\} > -4\} = Re\{s\} > -1$.

$$\rightarrow X(s) = \frac{1}{s+1} + \frac{1}{s+4} = \frac{2s+5}{s^2+5s+4}, \qquad Re\{s\} > -1$$